Continuous-Time Option Games Part 2:

Oligopoly, War of Attrition and Bargaining under Uncertainty

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Abstract

This sequel paper analyzes other selected methodologies and applications from the theory of continuous-time (real) *option games* – the combination of real options and game theory. In the first paper (Dias & Teixeira, 2003), we analyzed preemption and collusion models of duopoly under uncertainty. In this second paper we focus on models of oligopoly under uncertainty, war of attrition under uncertainty, and the changing the war of attrition game toward a bargaining game. In the oligopoly model we follow Grenadier (2002), discussing two important methodological insights that simplify many option games applications: the Leahy's *principle of optimality of myopic behavior* and the "artificial" perfectly competitive industry with a *modified demand function*. We discuss both the potential and the limitations of these insights. Next, we extend to the continuous-time framework the option game model presented in Dias (1997), a *war of attrition under uncertainty* applied to oil exploration prospects. In this model of positive externality the follower acts as free rider receiving additional information revealed by the leader's drilling outcome. The way to model the *information revelation* in oil exploration is another extension of the original option game model. In addition, we analyzed the possibility of *changing the game* with the oil firms playing the *bargaining game* be perfect Nash equilibrium. Cooperation can increase the value of the firms thanks to additional *private* information revelation provided by a contract. We quantify the degree of information revelation with the convenient learning measure named *expected variance reduction*. The bargaining game strategy must be compared mainly with the follower strategy in asymmetric war of attrition. We set the game threshold window where the bargaining alternative dominates any war of attrition outcome. We also show that the option game premium can be much higher than the traditional real option premium in either war of attrition or bargaining game. This is generally the opposite of the oligopoly under uncertainty case, when the option game premium is lower than the traditional option premium, is zero in the oligopoly limit of infinite firms, and can be even negative in special preemption cases.

Keywords: option games, option exercise games, real options, stochastic game theory, oligopoly under uncertainty, Leahy's optimality of myopic behavior, war of attrition, information revelation, changing the game, cooperative bargaining, option game premium.

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1 - Introduction

This paper is a sequel of our previous work (Dias & Teixeira, 2003) on continuous-time (real) option games. Option games models comprise the combination of two very important (Nobel laureate) and *complementary* theories, namely *options pricing* and *game theory*. Although discrete-time models are generally more intuitive, in most cases continuous-time models permit more general conclusions and more professional software. In our previous paper, after a short historic on option games literature, we focused on two alternative methodologies to solve preemption and collusion models of duopoly under uncertainty. In addition, we examined the role of mixed strategies in both symmetric and asymmetric duopolies.

In this second paper we focus mainly in two models. First, the oligopoly under uncertainty – with new artifices to simplify the solution of option games models. Second, the war of attrition under uncertainty – which can be viewed as the opposite to the preemption models. We also analyze the interesting possibility of changing the game from war of attrition to bargaining. We continue to highlight concepts, tools, and methodologies to solve option games rather than theoretical details.

In the oligopoly model we follow Grenadier (2002), which extends the classic paper of Leahy (1993) with his *principle of optimality of myopic behavior*. The applicability of this principle is well discussed in Dixit & Pindyck (1994, mainly chapter 9, section 1; but also chapters 8 and 11), but Grenadier extends this principle to oligopoly models. Perhaps his main contribution in this paper is the solution of oligopoly exercise strategies using an "artificial" perfectly competitive industry with a *modified demand function*. With these two artifices, we can solve many option games models using single agent's optimization procedures and the usual real options tools, without the necessity to use more complex techniques adopted in game theory like searching *fixed-points* from the players' best-response correspondences.

In the war of attrition under uncertainty model we extend the paper of Dias (1997), who worked with discrete-time option game model applied to oil exploration of two neighboring correlated prospects owned by two different oil companies. The option exercise is the drilling of one exploratory well – the *wildcat*, and part of the information revealed by the drilling is public so that the option exercise generates a positive externality that benefices the follower, who can decide about the option exercise with better information. So, there is a *second mover advantage* in contrast with the first mover advantage from the preemption models.

This paper is organized as follows. The section 2 presents the oligopoly under uncertainty model based in Grenadier (2002), but with some simulations and charts not presented there. Section 3 discusses the war of attrition under uncertainty applied to oil exploration, with discussion on information revelation modeling issues and equilibrium possibilities. Section 4 analyzes the "changing the game" alternative with the oil companies abandoning the war of attrition in favor of a cooperative bargaining game, with negotiation of rights and options. Section 5 presents some conclusions and suggestions for future research.

2 – Oligopoly under Uncertainty: The Grenadier's Approach

This section is based on Grenadier's model on oligopoly under uncertainty (Grenadier, 2002). For sake of space we present only selected results, but it includes some equilibria simulations with charts not presented in the original work. This addition is because we judge important to highlight important concepts such as the comparison between monopoly, duopoly, and oligopoly outputs for the same stochastic shock in the demand; and the concept of *upper reflecting barrier* limiting the maximum prices in oligopoly due to the (even imperfect) competition effect.

Grenadier (2002) has at least two very important contributions to the option-games literature:

- Extension of the Leahy's "Principle of Optimality of Myopic Behavior" to oligopoly; and
- The determination of oligopoly exercise strategies using an "artificial" perfectly competitive industry with a modified demand function.

Both insights simplify the problems solution because "*the exercise game can be solved as a single agent's optimization problem*" and the usual real options tools in continuous-time. In order to solve the option-game problem it is not necessary more complex techniques to search fixed-points from the players' best-response correspondences. We can even use Monte Carlo simulation of the stochastic demand to solve this model, as we will see later.

In the first insight, the myopic firm (denoted by i) is a firm that, when considering the optimal entry in a market, assumes that all the other firms production (denoted by Q_{-i}) will remain constant forever. As Dixit & Pindyck (1994, p.291) mention, "*each firm can make its entry decision ... as if it were the* *last firm that would enter this industry, and then making the standard option value calculation*" and "*it can be totally myopic in the matter of other firms' entry decisions*". The remarkable property of the optimality of myopic behavior was discovered by Leahy (1993, "Investment in Competitive Equilibrium: The Optimality of Myopic Behavior") and has been used and extended in many ways. See also Baldursson & Karatzas (1997).

The Grenadier's paper is closely related to Dixit & Pindyck (1994, mainly chapter 9, section 1; but also chapters 8 and 11). In Dixit & Pindyck (chapter 9) each firm produces only one unit so that the total industry output is the number of firms, whereas in Grenadier's model the number of firms is fixed (n) but each firm can add more than one unit of production. Perhaps the Grenadier's way to model oligopoly is more useful and realistic (an improvement over Dixit & Pindyck), e.g., monopoly, duopoly, and perfect competition are particular cases respectively for $n = 1, 2$, and ∞. However, for the asymmetric firms case, the unit production firm approach of Dixit & Pindyck has advantages over the Grenadier's way, because it is only an ordering problem (low-cost firms enter first).

However, in both cases is necessary to assume that the *investment is infinitely divisible* (firm i can add an infinitesimal capacity dq by an infinitesimal investment dK). Although it is more realistic the assumption of discrete-size (lump-sum) additions of capacity by the firms, the approach may be a reasonable approximation in many industries (e.g., new investment is a small fraction of current industry capacity), mainly if the aim is the *industry equilibrium study*. But the model is less realistic at *firm-level decision*. This necessary approximation allows the extension of the Leahy's principle of optimality of myopic behavior, which simplifies a lot the problem solution. However, this assumption is not necessary for the *perfectly competitive case* of Leahy (see also the wonderful explanation of Dixit & Pindyck, chapter 8, section 2), where the competitive firm analyzes myopically a *lump-sum* investment to enter in the competitive industry.

The second Grenadier's insight permits the application of the important results obtained from the perfectly competitive framework into the apparently more complex case of imperfect competition of dynamic oligopoly under uncertainty. As example, Grenadier presents an extension of his previous paper on real-estate markets that considers the *time-to-build* feature for a perfectly competitive industry (Grenadier, 2000). He obtained simple closed-form solutions for the equilibrium investment strategies using this smart artifice. Other results obtained for perfectly competitive markets could also be easily extended to the oligopoly case. Examples are the results from Lucas & Prescott (1971) on rational expectations equilibrium, Dixit (1989) on hysteresis models, and Dixit (1991) for price ceilings models, among other known results.

Let us describe the model. Assume that each firm from the n-firms oligopoly holds a sequence of investment opportunities that are like compound perpetual American call options over a production project of capacity addition. The first assumption is that all firms are equal, with technology to produce a specific product. The output is infinitely divisible, and the unity price of this product is P(t). This price changes with the time because the *demand* D[X(t), Q(t)] evolves as continuous-time stochastic process. Assume either that the firms are risk-neutral or that the stochastic process X(t) is risk-neutral (that is, the drift is a risk-neutral drift = real drift less the *risk-premium*).

Initially, as in Grenadier's paper, let us consider a more general diffusion process and a more general inverse demand function, given respectively by:

$$\mathbf{dX = \alpha(X)\, dt + \sigma(X)\, dz} \qquad \textbf{(1)}$$

$$\mathbf{P(t) = D[X(t), Q(t)]} \qquad \textbf{(2)}$$

For the popular geometric Brownian motion (GBM), just make $\alpha(X) = \alpha \cdot X$; and $\sigma(X) = \sigma \cdot X$. As usual, α is the (real) drift, σ is the volatility, and dz is the Wiener increment.

In the Cournot-Nash perfect equilibrium, strategies are quantities and the market clears the price at each state of the demand along the time. Firms choose quantities $q_i^*(t)$, i = 1, 2, ... n, maximizing its payoffs and considering the competitors best response q_{-i}^*.

With the simplifying assumption of equal firms, the natural consequence is the choice of <u>symmetric</u> Nash equilibrium, that is, $q_i^*(t) = q_j^*(t)$ for all i, j. Denote the total industry output in equilibrium by Q*(t). The optimal output for each firm from this n-firm symmetric oligopoly Nash equilibrium is:

$$\mathbf{q_i^*(t) = Q^*(t) / n}$$

The exercise price of the option to add a capacity increment of dq is the investment **I . dq**, where I is the unitary investment cost, equal for all firms. The option to add capacity is exercised by firm i when the demand shock X(t) reaches a threshold level $X_i^*(q_i, Q_{-i})$.

Grenadier summarizes the equilibrium in his Proposition 1, with a partial differential equation (PDE) and three boundary conditions. The PDE is obtained using the standard option pricing approach (Itô's Lemma, risk-free portfolio, etc.). The first and second boundary conditions are the value-matching and smooth-pasting conditions, as usual in continuous-time real options framework. However, the third condition is the strategic one, requiring that each firm i is maximizing its value $V_i(X, q_i, Q_{-i})$ given the competitors' strategies (thresholds).

The third condition is a value-matching at the competitors' threshold $X_{-i}(q_i, Q_{-i})^*$, which is equal to $X_i(q_i, Q_{-i})^*$ due to the symmetric equilibrium. The third condition is also like a *fixed-point* search over the best response maps. However, this condition will not be necessary with the Grenadier's Proposition 2, extending the myopic optimality concept to oligopolies. Proposition 2 assumes that *investment is infinitely divisible* (see discussion above) and tells that the myopic firm threshold is equal to the firm's strategic (Cournot-Nash perfect equilibrium) threshold. Proposition 3 will set the main equilibrium parameters with only two boundary conditions.

Denote the value of myopic firm by $M^i(X, q_i, Q_{-i})$. Let us to work with the value of a myopic firm's marginal output $m^i(X, q_i, Q_{-i})$ defined by:

$$\mathbf{m^i(X, q_i, Q_{-i}) = \partial M^i(X, q_i, Q_{-i}) / \partial q_i} \qquad \textbf{(3)}$$

Given the symmetry, we can write $X^i(q_i, Q_{-i})^* = X^*(Q)$ because $\mathbf{q_i = Q/n}$ and $\mathbf{Q_{-i} = (n - 1) . Q / n}$. His proposition 3 establishes the symmetric Nash equilibrium: each firm will exercise its investment option whenever X(t) rises to the trigger X*(Q). Let **m(X, Q)** denote the *value of a myopic firm's marginal investment*. The following PDE and two boundary conditions determine both X*(Q) and m(X, Q):

$$\mathbf{0.5\, \sigma(X)^2\, m_{XX} + \alpha(X)\, m_X - r\, m + D(X, Q) + (Q / n)\, D_Q(X, Q) = 0} \qquad \textbf{(4)}$$

Subject to:

$$\mathbf{m[X^*(Q), Q] = I} \qquad \textbf{(5)}$$

$$\mathbf{\partial m[X^*(Q), Q] / \partial X = 0} \qquad \textbf{(6)}$$

Where the subscripts in the PDE (eq. 4) denote partial derivatives, the equation (5) is the value-matching at X*(Q), and equation (6) is the smooth-pasting condition. The last two terms in the right side of equation (4) comprise the non-homogeneous part of the PDE, the called "cash-flow" terms. This non-homogeneous part will play a very special role in Grenadier's paper, because it is the ***modified demand function*** mentioned early. The first three terms of the PDE comprise the homogeneous part of the PDE. It is very known in real options literature (Dixit & Pindyck, 1994).

The nice issue is that only two "real options" boundary conditions at the common threshold level X*(Q) are *sufficient* for the optimal strategic exercise of the option due to his Proposition 2, which says that the myopic firm threshold is equal to the firm's strategic threshold.

Grenadier (section 5) shows that, besides the monopoly and perfectly competitive industry cases, it is also possible to solve the oligopoly case as a *single agent optimization problem*. The procedure is just to pretend that the industry is perfectly competitive, maximizing a "fictitious" objective function. This "fictitious" objective function uses an ***"artificial" demand function*** defined by:

$$\mathbf{D'(X, Q) = D(X, Q) + (Q / n)\, D_Q(X, Q)} \qquad \mathbf{(7)}$$

As mentioned in the introduction, this is a very important result because permits the extension of known (or easier to obtain) results in perfectly competitive setting to the oligopoly case. In section 6, Grenadier shows the equilibrium with time-to-build as example of this extension. Here we focus in the example from his section 3 but with some further simulations not showed in the paper.

Consider a specific diffusion process – geometric Brownian motion, and also a specific inverse demand function – a multiplicative shock constant-elasticity demand curve, given respectively by:

$$\mathbf{dX = \alpha\, X\, dt + \sigma\, X\, dz} \qquad \mathbf{(8)}$$

$$\mathbf{P(t) = X(t)\, .\, Q(t)^{-1/\gamma}} \qquad (9)$$

Where $\gamma > \mathbf{1/n}$ ensures that marginal profits are increasing in X. Assume also that the risk-free discount rate is strictly higher than the drift[1] α. The optimal threshold X*(Q) is given by:

$$\mathbf{X^*(Q) = v_n\, .\, Q^{1/\gamma}} \qquad \mathbf{(10)}$$

[1] For the risk-neutral drift, which for the GBM is equal to $\alpha' = r - \delta$, just assume that the dividend yield $\delta > 0$.

Where $\mathbf{v_n}$ is an ***upper reflecting barrier***, that is, the maximum price that the product can reach in the oligopolistic market. When the price reaches this level, firms add capacity in a quantity so that the price is reflected-down due to the additional supply.

For this multiplicative demand shock, while X(t) follows the (unrestraint) GBM, the price P(t) follows a constrained GBM with upper reflecting barrier v_n, given by:

$$\mathbf{v_n = \left(\frac{\beta_1}{\beta_1 - 1}\right)\left(\frac{1}{1 - 1 / n\,\gamma}\right)(r - \alpha)\, I} \qquad \textbf{(11)}$$

Where $\beta_1 > 1$ is the known positive root of the quadratic equation: $0.5\ \sigma^2\ \beta\ (\beta - 1) + \alpha\ \beta - r\ =\ 0$. Note that the threshold X*(Q) is decreasing with the number of firms in the oligopoly (n), which looks intuitive. It is the competitive effect with intensity n, reducing the entry threshold.

In order to keep the prices at or below v_n, the addition of capacity dQ (= n dq) when X(t) > X*(Q), with cost I dQ, is larger as larger is the difference X(t) – X*(Q). In other words, if X(t) > X*(Q) then $Q(t) = (X(t) / v_n)\gamma$.

What is the ***option premium*** when exercising this strategic option in the n-firms oligopoly? Grenadier defines this option premium as the NPV at X* per unit of investment I, OP(n) given by:

$$\mathbf{OP(n)\ =\ 1 / [(n\,\gamma) - 1]} \qquad \textbf{(12)}$$

Hence, when n tends to infinite the OP(n) tends to zero, a consistent result. See in Dixit & Pindyck (1994, chapter 8) that the NPV is zero for the perfectly competitive case. For n finite the NPV is positive but as small as large is the number of firms (n), i.e. as intense is the competition. In Dias & Teixeira (2003) we saw that in a special case, the NPV of exercising an expansion option could even be negative, in order to avoid the competitor entry that could be even worse for the current firm operations. In the next section we will see the opposite: for war of attrition the option premium from optimal exercise can be even higher than the traditional (monopolistic) real option premium.

Let us perform some numerical calculations in order to see the power of the above concepts to understand the oligopoly equilibrium under uncertainty. We use the same numerical values adopted in Grenadier's paper, section 3 for his figure 1, except where indicated. The values are: $\alpha = 0.02$ p.a.; $r = 0.05$ p.a.; $\sigma = 0.175$ p.a.; $\gamma = 1.5$; $n = 10$ firms; $I = 1$ \$; $Q(0) = 100$ units; and $X(0) = 1.74$ \$/unit.

An interesting and practical feature of the *principle of optimality of myopic threshold* is that we can use a Monte Carlo simulation in order to solve the model. It is not necessary to work backwards because we know the ("myopic") threshold level X*(Q(t)) in advance. So, if this threshold is triggered by the simulated sample-path of the demand X(t), new capacity is added by the oligopolistic firms and it is easy to study many properties of the strategic exercise of options in an oligopoly and the aggregate behavior of the industry in the long-run, such as the industry output Q(t), the investment along the years, the prices evolution, etc. **Figure 1** shows some of these features for 10-firms oligopoly case presenting a certain demand sample-path X(t) over 10 years.

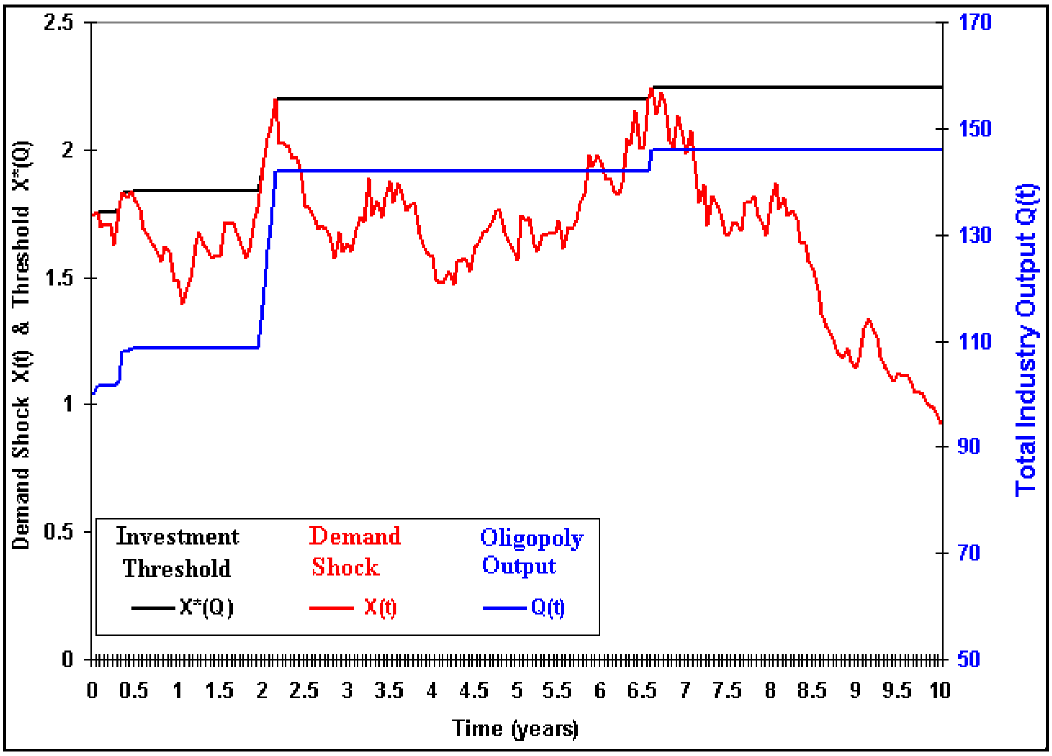

Figure 1 – Demand Sample-Path and Strategic Exercise in 10-Firms Oligopoly

When the demand rises at the threshold level all firms exercise options to expand capacity, increasing the aggregated industry output. In this model, the firms' addition of capacity is proportional to the difference between the demand shock X(t) and the threshold level X*(Q(t)), if positive. In case of the demand below the threshold X*, no investment is performed (and no exit as well). In the Grenadier's model the firms are equals, so that in 10-firms oligopoly case each firm adds 1/10 of the new capacity Q(t) – Q(t – dt) in case of positive shock at t, if X(t) > X*(Q(t – dt)).

Figure 1 also shows that, for this specific sample-path, after the year 8 the demand drops to levels well below the demand level at t = 0, but the total industry output remains (so that the prices drops). This model does not consider reduction of industry output due to low demand state. A possible improvement in the model is to consider other options like the option to temporary stopping (at cost) and the option to exit (or at least the option to contract).

Figure 2 shows for one demand evolution sample-path, that the industry total output Q(t) is much higher for the 10-firms oligopoly (n = 10) case than for duopoly (n = 2), which presents a higher industry output than the monopoly case (n = 1).

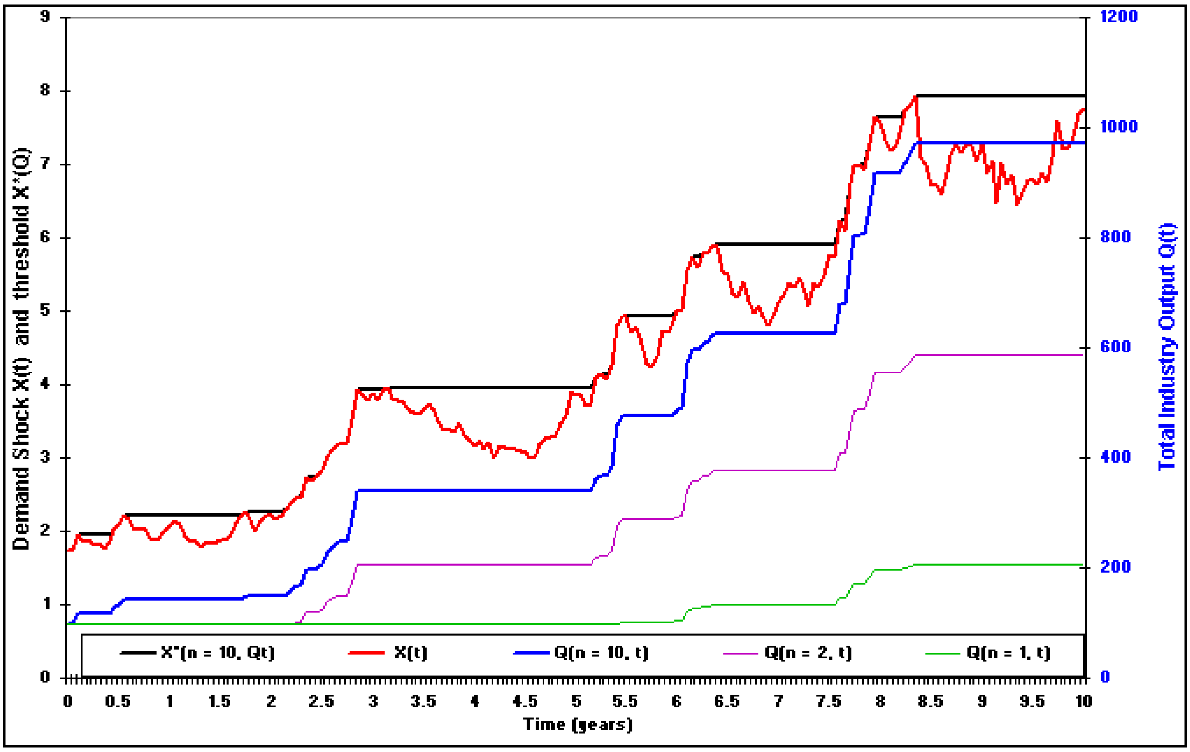

Figure 2 – Industry Output under Monopoly, Duopoly, and 10-Firms Oligopoly

Figure 2 shows that, for the same demand evolution, after 10 years the oligopoly with 10 firms produce together near 1000 units, the duopoly produces near 600 units, and the monopoly produces 200 units (only 1/5 of the 10-firms oligopoly).

Figure 3 below shows the evolution of the prices, considering a possible demand sample-path with the respective oligopoly capacity evolution. Note that there is an ***upper reflecting barrier*** at \$0.8081/10 units, so that when exists a positive demand shock reaching this reflecting barrier, the oligopoly addition of capacity is sufficiently high to the prices either to remain at this level (if demand remains rising) or to be reflected-down.

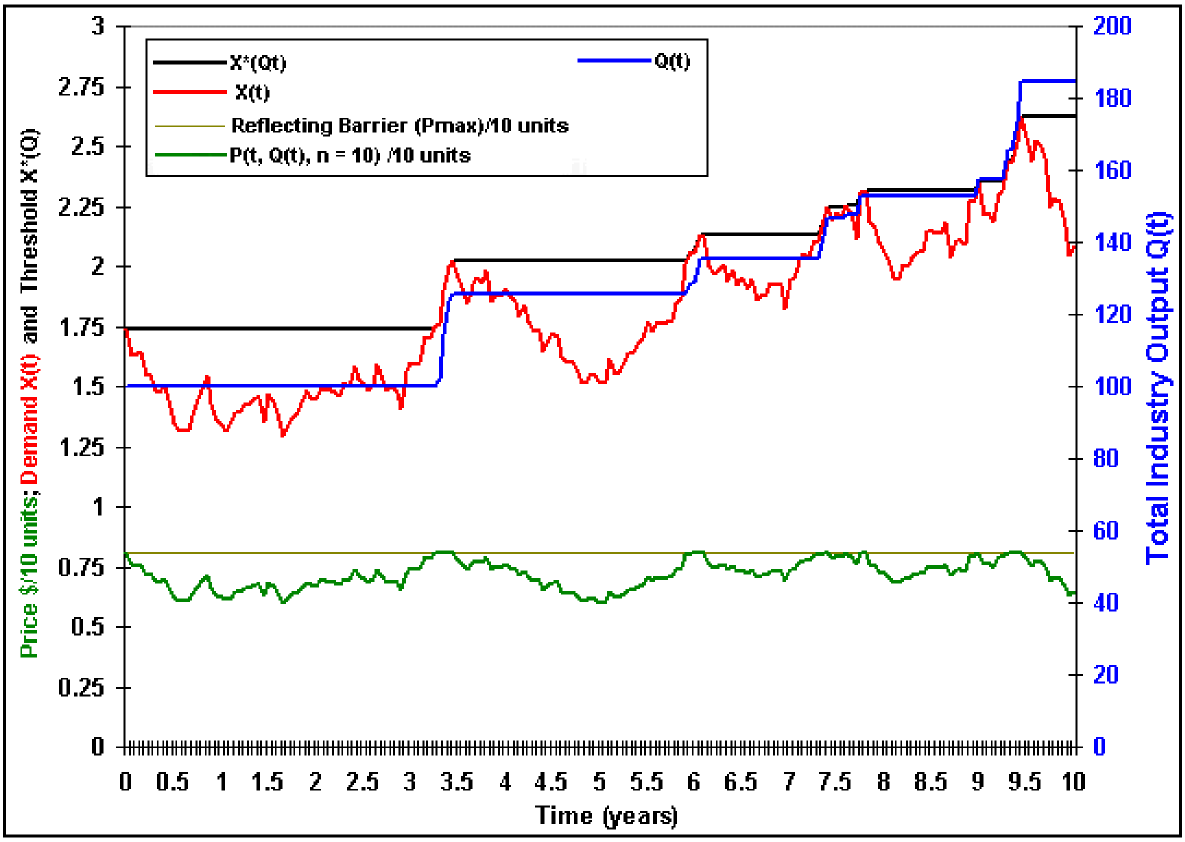

Figure 3 – Price Evolution with Demand and Industry Output for 10-Firms Oligopoly

3 – War of Attrition under Uncertainty

In this section we discuss an important class of games named war of attrition. First, we briefly present a more static variant of this *waiting game* known as *chicken*[2]. Two teenagers drive their cars toward each other. The first to deviate (avoiding a collision) is the "chicken" and loses the game[3]. Here, the first to exercise the option (the "leader" L)[4] loses the game so that the follower (F) role is more valuable (F > L). The simultaneous exercise (S) is worse than the follower and equal or higher than the leader playing, i.e., F > S ≥ L. However, both players prefer to exercise the option (chicken) than the situation of *both* players never exercising the option ("car collision"), i.e., the simultaneous

[2] This game is well discussed in Dixit & Skeath (1999, pp. 9, 110-112, 136-140, 331-334). It is more static because is commonly analyzed using only a strategic form matrix game, without specifying the *stopping time* strategies. However, some authors (e.g., Fudenberg & Tirole, 1991, p.119 n.7) consider chicken just another name for the war of attrition.
[3] Another version for the game of chicken is showed in the movie *Rebel without Cause* with James Dean.
[4] We use the terms leader and follower in order to compare with the preemption case described in Dias & Teixeira (2003).

waiting (W) strategy has the worst payoff W < L. With these payoffs ordering we will identify all the Nash equilibria. Figure 4 shows the game of chicken in strategic form.

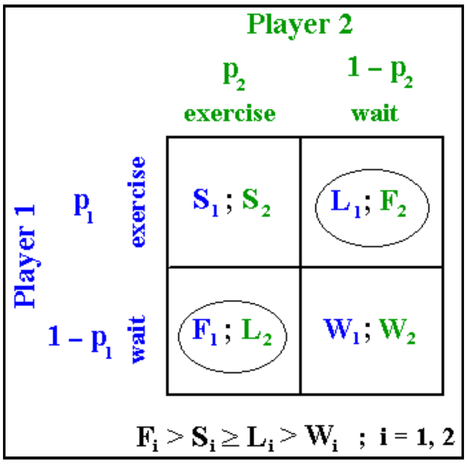

Figure 4 – The Game of Chicken

The two pure-strategy Nash equilibria are (F_1 ; L_2) and (L_1 ; F_2). There is also one mixed-strategy Nash equilibrium that is a randomization over the two pure-strategy Nash equilibria, with each player i = 1, 2, choosing the option exercise probability p_i so that it *keeps the opponent j indifferent* between exercising or not the option[5]. This probability is function of the *opponent* payoffs F_j, S_j, L_j, and W_j:

$$p_i = \frac{L_j - W_j}{L_j - W_j + F_j - S_j} \tag{13}$$

This probability is both strictly lower than 1 and strictly higher than 0 because $F_j > S_j$ and $L_j > W_j$. Note that we obtain these <u>three equilibria</u> if either $S_j = L_j$ or $S_j > L_j$ (both are games of chicken). However, if either $F_j = S_j$ or $L_j = W_j$ we get only <u>one</u> degenerate mixed strategy equilibrium. This result explains why we exclude the case of $F_j = S_j$ in this game[6]. Note also that if $L_j >> W_j$, the exercise probability p_i is near to 1, i.e. even with our high probability of exercise, the opponent will be indifferent on exercising or not the option because W is very low (fear of collision).

Consider that the players are equal (same payoffs). The only *symmetric* equilibrium is the mixed strategy one ($p_i = p_j$, not necessarily 50%). *Evolutionary game theory* provides an important rationale for selecting one of these multiple (three) Nash equilibria, by choosing the one that is also an

[5] In fact the players *don't want* to keep the opponent indifferent. This is just a known mixed-strategy rule of thumb that results from the payoff maximization problem solved by each player considering the opponent best responses.

evolutionary stable strategy (ESS)[7]. In the game of chicken the only ESS is the mixed strategy Nash equilibrium[8]. ESS's are strategies *dynamically stable* in the sense that are able to resist the infiltration of mutant (alternative) strategies. Hence, ESS's are more probable outcomes in the long run. In addition, in two-players games, a regular ESS is also *subgame perfect equilibrium* (SPE) because an ESS cannot involve a weakly dominated strategy. But even in finite games the existence of ESS is not guaranteed, whereas at least one SPE always exists. ESS is a more robust condition than the SPE.

In stochastic games, the introduction of a state variable following a stochastic process (like the oil prices P), enhance the problem of multiplicity of equilibria in war of attrition games. So, as usual in stochastic games, we limit our focus on the *Markov equilibria*, i.e. equilibria that are function of the current state only[9], which follows a Markov process. Markov equilibriums are also subgame perfect. We want identify at least one *Markov perfect equilibrium* (MPE) and, if possible, the MPEs that are also evolutionary stable.

War of attrition was analyzed first by Maynard Smith (1974) in a game between animals fighting for a prize (a territory or a prey). There is a cost to remain fighting, and this cost is increasing with the time length. So, the value of *never stopping* (or even later stopping) for both players is lower than the value of one player conceding (stopping) immediately. In this game, if one animal "stops" it concedes the prize generating the positive externality to the other player. It is a "waiting" premium, so that the follower value is higher than the leader value in the war of attrition. In the unlikely case of simultaneous option exercise (both animals leaving the combat at the same time) either the contest is decided randomly (Maynard Smith, 1982, but the expected gain is lower than the follower gain, i.e. $F(t) > S(t) > L(t)$) or neither wins the prize ($F(t) > S(t) = L(t)$, as in Fudenberg & Tirole, 1991, p.119)[10]. Simultaneous exercise in war of attrition is always less valuable than the follower value.

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[6] There are some games of positive externalities with *network effects* which $F_j = S_j$ or even $F_j < S_j$. However, they belong to other class of games, with very weak relation with our focus, i.e. the games of chicken and war of attrition.

[7] ESS is the key equilibrium concept in evolutionary game theory due to Maynard Smith & Price (1973) and was born almost simultaneously with the analysis of war of attrition games (Maynard Smith, 1974). See also Maynard Smith (1982) and Hammerstein & Selten (1994). There is a growing literature in economics with ESS applications.

[8] A mixed strategy σ is ESS if and only if: (a) It is a best reply to itself; and (b) for any alternative (*mutant*) best reply σ' to σ, it does better than σ' does against itself, i.e. for all the available strategies $\sigma' \neq \sigma$ with payoffs $\pi_1(\sigma', \sigma) = \pi_1(\sigma, \sigma)$, we must have $\pi_1(\sigma, \sigma') > \pi_1(\sigma', \sigma')$. The condition (a) simply tells that ESS must be Nash equilibrium (NE) with itself, and the condition (b) is the *stability condition* against the invasion of mutant strategies. Maynard Smith (1982, appendix B) showed that a matrix game with two pure NE strategies always has an ESS and proved that our equation (13) is ESS.

[9] The current state variable summarizes the direct effect of the past on the current game, see e.g., Fudenberg & Tirole (1991, chapter 13). See also Kapur (1995) for Markov perfect equilibria in war of attrition games.

[10] However, Huisman (2001, pp.100-101) included as war of attrition the case of $L(t) > S(T)$, with $t \in [0, T]$. This doesn't agree with the usual war of attrition definitions from Fudenberg & Tirole (1991) and Maynard Smith (1982).

War of attrition belongs to the general class of *timing games* or *optimal stopping games*, i.e. games where the players' pure strategies are stopping times choices. So, at each moment the set of actions for each firm is $A_i(t)$ = {stop; don't stop}, see e.g., Fudenberg & Tirole (1991, p.117). Here stopping time means time to stop the "wait and see" policy by exercising the real option. These types of games can be classified into two categories: games of *negative externalities* (e.g., preemption games) and games of *positive externalities*. The latter includes models of *war of attrition* and *network games*. Contrasting with the war of attrition, in network games the simultaneous investment option exercise strategy (S) is more valuable than single investment (e.g., adoption of a standard in new technology market), so that the simultaneous exercise value is not worse than the follower value, and generally is even more valuable. Huisman (2001, pp.205-208) analyzed this important option-game, but not the war of attrition as defined by Fudenberg & Tirole (1991) or Maynard Smith (1982).

The war of attrition game has many applications in economics. In industrial economics literature an example is the *abandon option* exercise in duopolies of *declining industries*. In this kind of industry the exit of one firm benefits the other firm because the remaining firm becomes monopolistic, getting an additional profitable life (Ghemawat & Nalebuff, 1985). Fudenberg & Tirole (1986) analyzed the same case but with focus on *nondeclining* industry. Another example is in *noncooperative bargaining games*; see for example Ordover & Rubinstein (1986). Noncooperative bargaining games are wars of attrition in the sense that players are impatient (there is a cost to delay the deal), proposals are fixed (the prize is the difference between the proposals) but agreement requires approval from both players, so that one player must concede (stop) to reach agreement.

An interesting war of attrition application arises in oil exploration because the positive externality is subtler: the option exercise by the leader generates an *information revelation* that benefits the other player (the follower), which can use this information as a *free rider* to decide about its option exercise itself. In the context of traditional game theory, this case has been object of research mainly by Hendricks (e.g., see Hendricks & Porter, 1996, or Hendricks & Wilson, 1985), whereas in the option-games context Dias (1997) analyzed this game in discrete time using a game tree.

In order to analyze the oil exploration case we will set some formalization. Following Fudenberg & Tirole (1991, p.118), pure strategies s_i in war of attrition are *stopping times*[11], i.e., simple maps from the set of dates to the set of feasible actions {stop; don't stop}. The game is over when at least one

[11]In a more general case, pure strategies are *stopping sets* (time intervals in which stopping is optimal).

player stops (exercise the option by drilling the exploratory well). Let us consider the case of two players, i and j. The two-player case is important in exploration business practice because is more common[12], and is important for the theory because any n-player analysis needs start backwards with a subgame in which only 2 players remain. For the player i denote the leader value by $L_i(t)$, the follower value by $F_i(t)$, and the value of simultaneous exercise by $S_i(t)$, according the stopping times:

- Value of player i $= L_i(t)$ if $t_i < t_j$
- Value of player i $= F_i(t)$ if $t_i > t_j$
- Value of player i $= S_i(t)$ if $t_i = t_j$

We will allow the game be *finite*, because the oil companies have a finite lived real option that is the right to drill a prospect with some probabilities to find oil reserves during a fixed contractual time[13]. Note that the perpetual option case is easier than the finite lived option because in the latter case the time is a *state variable*, demanding numerical methods to solve the problem. However, finite games permit easier application of backward induction argument in equilibrium analysis. Let the game be defined in the interval [0, T]. We are interested in war of attrition without network effects because the players cannot capture the premium (better informed decision) either being leader or exercising the option simultaneously[14]. We characterize the *war of attrition* with the following conditions:

(a) $F_i(t) > L_i(t)$ for $t \in [0, T)$
(b) $L_i(t) = S_i(t)$ for $t \in [0, T]$
(c) $L_i(t) \downarrow t$ for $t \in (0, T)$

Condition (a) tells that the positive externality has value so that the follower value is higher than the leader value. Condition (b) set that simultaneous exercise does not generate externality gains for the players. Condition (c) tells that is better to be leader earlier than later. Condition (c) is *ceteris paribus*, i.e., for the same market conditions (of course the value of be leader rises if the oil price

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[12] The information revelation from neighboring tracts (with prospects in the same geologic play) is stronger or much stronger than from more distant areas. So, in most cases only two adjacent tracts have relevant information revelation to generate strategic interaction in practice. However, there are interesting cases with n > 2 players that can be analyzed with a similar methodology presented in this paper, but considering the sequential information revelation process.

[13] Other examples of finite war of attrition in economics: (a) in labor x management negotiations, the contract expiration can be a deadline for the game; (b) in a contract negotiation with a supplier, the date that the firm expects to run out of inventories is a deadline for the game. See Ponsati (1995) for other insights on finite war of attrition games.

[14] Other example of war of attrition *without network effects* is in price wars in declining industries: the premium (lower competition) is not captured either if the firm exercises the option to abandon alone or simultaneously.

rises over time). In this option game, the cost to postpone the exercise of a *deep-in-the-money*[15] real option penalizes the waiting strategies.

In our oil exploration game, if at the expiration the game is still a war of attrition, we will set additionally that $F_i(T) = L_i(T)$, so that at the option expiration the leader value is equal to the follower value. The reason is that the follower cannot use the information revealed later because the informed follower's option to drill the wildcat has already expired. It is also true even if T is infinite (perpetual option): due to the discounting effect (and considering finite payoffs) we have $F_i(\infty) = L_i(\infty) = 0$.

As usual in the literature, for didactic reasons we assume that the wildcat drilling is instantaneous. In reality it takes about three months to reveal information on the existence of commercial oil reserves. So, if firm j exercises the option (leader) putting a rig over its prospect area, the follower firm (i) needs about three months to know the outcome. There is a *time-to-learn* or a *revelation time* (t_R). If you consider the time-to-learn effect, the follower value will be penalized when compared with our simplified approach of setting $t_R = 0$. This practical issue can be analyzed in details in a future work, but initial simulations (Dias, 2004) shows that the main effect when considering $t_R > 0$ are: (a) the informed follower value is lower; (b) the threshold for optimal simultaneous investment is lower because this threshold is the point where the value of leader and follower are equal; (c) there exists a P' in which for $P \geq P'$ the unique Markov perfect equilibrium (MPE) is the simultaneous exercise[16].

One important difference of this paper in relation to Hendricks & Porter (1996) is the oil exploration <u>information revelation modeling</u>. They model the discovery oil <u>volume</u> in the ground as a lognormal distribution with expected value being revealed with the adjacent tract drilling. However, the public information revealed with the neighboring tract drilling has important impact on the *chance factor* (probability of success), sometimes some impact in expected petroleum quality, and very poor or no impact in the expected petroleum volume of adjacent non-drilled prospects. Seismic surveys performed before the wildcat drilling indicate mainly the size of the structures. So, imagine that seismic information points two prospects, one in each adjacent tract, being one a big structure and the other a small structure. If the bigger structure is drilled first and find an oil deposit, this positive outcome (but not the geological details) becomes public information so that the chance factor for the

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[15] An American call option (like our real option) is deep-in-the-money if it is optimal the immediate exercise.

[16] For $t_R = 0$, if the simultaneous exercise is optimum, the informed follower value is equal to the simultaneous value and we have three MPEs: simultaneous exercise; firm i as leader/firm j as follower; and firm j as leader/firm i as follower. But we can set t_R arbitrary small and > 0, in order to rule out the other equilibria and to endorse the condition $F_i(T) = L_i(T)$.

adjacent tract is revised upward. But the expected tract volume continues being smaller than the adjacent one, so that only the chance factor is updated with the adjacent positive revelation[17].

In order to understand better the last point, we need to discuss deeper the chance factor[18]. Chance factor to find a petroleum reserve can be viewed as the product of 6 independent chance factors: 1) probability of *generator rock* existence; 2) probability of *seal rock* existence; 3) probability of *reservoir rock* existence; 4) probability of *geologic fail* existence linking the generator with the reservoir rock; 5) probability of a *geometric trap* existence (seal/reservoir rock geometry); and 6) probability of *geological synchronism* occurrence (geologic timing coincidences when generating, moving, storing, and trapping oil). The seismic survey (specially the 3-D seismic) gives good indications about the rocks existence and structural aspects in general, but virtually no (or very poor) indication about the fluids in the reservoir rock (if oil or water). Hence, seismic surveys give good indications for the first 5 factors listed above, but almost nothing about the last one (synchronism). Neighboring drilling outcomes in the same geologic play reveal mainly (and strongly) the last factor, so that adjacent drilling complements the seismic information[19]. However, even with both sources of information some uncertainty remains in our prospect. We develop this point later in the text.

The game is solved backwards, as standard in timing games and in real options. We need to know the ending payoffs for the strategies. Consider the follower with the additional information revealed by the leader drilling. The follower will revise the expectations on the chance factor and EMV, checking out if the option to drill is deep-in-the-money or not. So, the follower problem is a pure real options case because there is no strategic interaction anymore after the first exercise.

Imagine that the follower drills the wildcat. In case of success (confirming the existence of oil reserve), the follower has an *option to develop* the oilfield. So, there is a compound real option: the option to drill the wildcat gives – in case of success, the option to develop the oilfield[20]. Of course a *necessary* condition for the exploratory option be exercised is that in case of success the option to

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[17] However, some nonpublic information – mainly technical details like the oil-water contact, quality of rock and fluids in details, etc. – could update neighboring tracts beliefs on volumes and quality. But these details are not public, is necessary a partnership for the neighbor firm to have access to these details. We examine later this strong incentive for cooperation.

[18] We thank to Paulo Johann, Petrobras' Senior Consultant in geophysics, for discussions on chance factor and on the information revelation comparison between seismic survey and adjacent tract drilling outcome.

[19] So, seismic and adjacent drilling provide *complementary* information not *substitute* information as claimed by Hendricks & Porter (1988, p. 866).

[20] In a more general case we could consider the appraisal phase, with the follower drilling additional wells to get additional information about volume and quality of this reserve. However, in order to focus on option-game issues we consider only two sequential options, with the appraisal wells cost being included in the investment I_D.

develop be deep-in-the-money. There is no sense to spend earlier I_w and – in case of success, to keep the project idle for some time[21] (see a similar situation in Dixit & Pindyck, 1994, p.190)[22]. However, the development option can be deep-in-the-money but not the exploratory option.

We exercise the development option by paying the development investment I_D in order to receive the developed reserve asset V, so that we get the development net present value NPV = V – I_D with this option exercise. The value of the developed reserve is at least function of the long-run expected oil prices[23] (P), reserve volume (B, as the number of barrels), and the reserve quality (q). Let us consider a simple *parametric model* for V(P, B, q) named "Business Model"[24] in order to work our examples. In this parametric model, the NPV obtained with the development option exercise is:

$$\mathbf{NPV = q\ B\ P - I_D} \tag{14}$$

Denote the real option value R(P, t) to develop the oilfield as function of the state variables oil price (P) and time (t). Assume that the long-run expected oil prices follow a geometric Brownian motion[25] (GBM). Assuming complete markets for P and using the *contingent claims* method (see, e.g. Dixit & Pindyck, 1994, especially chapter 6, section 1D), we obtain the following *partial differential equation* (PDE) for the development option R(P, t).

$$\frac{1}{2}\sigma^2 P^2 \frac{\partial^2 R}{\partial P^2} + (r - \delta)\, P \frac{\partial R}{\partial P} - r\,R + \frac{\partial R}{\partial t} = 0 \tag{15}$$

Where σ is the oil price volatility, δ the oil convenience yield, and r the risk-free interest rate. The four boundary conditions for the PDE (eq. 15) are:

- If P = 0, $\mathbf{R(0, t) = 0}$ **(16a)**

[21] Other way to see this point: if our *underlying asset* is an option – and option asset has δ = 0, it is never optimal to exercise *earlier* the option on this option (known American option property). However, when this underlying option becomes deep-in-the-money, earlier exercise can be optimal because the underlying asset generates cash flow (δ > 0).
[22] With the assumption of instantaneous drilling we are neglecting the complications during the option exercise, e.g. during this drilling time the price can drops below the development threshold and we can wait after the wildcat drilling.
[23] Especially for large E&P projects, the oil price used by oil companies in discounted cash flow valuation is the middle to long-run *expectation* rather than the spot price due to the projects' long maturity. Typically (offshore case) there is a *time-to-build* of three years, and *payback* of more than 5 years.
[24] See a detailed discussion of this and alternative payoff models at www.puc-rio.br/marco.ind/payoff_model.html
[25] This expectation can be linked to both spot and futures prices and here is assumed known today and uncertain in the future. We assume that P follows a GBM with low volatility (futures price volatility can be used as proxy). An alternative using spot prices in a nonparametric (explicit cash-flow) model, shall consider more complex stochastic processes with mean-reversion towards a (perhaps stochastic, GBM) long-run level, and possibly considering jumps in spot prices.

- If t = T, $\mathbf{R(P, T) = max(q\ B\ P - I_D, 0)}$ **(16b)**

- If P = P*, $\mathbf{R(P^*, t) = q\ B\ P^* - I_D}$ **(16c)**

- If P = P*, $\dfrac{\partial \mathbf{R(P^*, t)}}{\partial \mathbf{P}} = \mathbf{q\ B}$ **(16d)**

These conditions are standard in real options literature[26]. We solve this real options problem with numerical methods like finite differences or analytical approximations, obtaining both the option value R(P, t) and the optimal decision rule (threshold curve P*(t)). Although exercising the exploratory options in different circumstances, the methodology presented for the development option is valid to both the leader and the follower. However, in the asymmetric case the values for the expected oil reserve parameters (q, B, I_D) can be different and we will use an additional subscript (i or j) when convenient to distinguish the players' payoffs and options.

Note that we obtain the development option only if we exercise the exploratory option (exercise price is the wildcat drilling investment I_W) <u>and</u> we have success (finding out undeveloped reserves). Denote the exploratory option value E(P, t) to drill the exploratory well (wildcat) as function of the state variables oil price (P) and time (t). Again using the contingent claims method, we obtain a similar partial differential equation (PDE) for the <u>exploratory option</u> E(P, t).

$$\frac{1}{2}\sigma^2 P^2 \frac{\partial^2 E}{\partial P^2} + (r - \delta) P \frac{\partial E}{\partial P} - r E + \frac{\partial E}{\partial t} = 0 \qquad \textbf{(17)}$$

Similarly, the four boundary conditions for the PDE (eq. 17) are:

- If P = 0, $\mathbf{E(0, t) = 0}$ **(18a)**

- If t = T, $\mathbf{E(P, T) = max[- I_W + CF\ (q\ B\ P - I_D), 0]}$ **(18b)**

- If P = P**, $\mathbf{E(P^{**}, t) = - I_W + CF\ (q\ B\ P^{**} - I_D)}$ **(18c)**

- If P = P**, $\dfrac{\partial \mathbf{E(P^{**}, t)}}{\partial \mathbf{P}} = \mathbf{CF\ q\ B}$ **(18d)**

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[26] Eq. 16a is a trivial condition; eq.16b is the condition at the legal expiration; eq.16c is the *value-matching* condition at the threshold level (P*); and eq.16d is the *smooth-pasting* condition at P*.

As before, we solve this real options problem with numerical methods like finite differences or analytical approximations, in order to find out both the options value surface R(P, t) and the threshold curve P**(t). For while we are considering only the real options problem, not the strategic iteration. If the oil price is below the threshold P**, the firm will optimally wait independently of the possibility of information revelation, i.e. independently of the war of attrition game. This game really starts when at least one prospect is deep-in-the-money so that really there is a "fighting cost" namely the cost to delay the exercise of a deep-in-the-money option. Hence, this argument points out that the leader threshold P_L cannot be lower than the "pure" exploratory option threshold P** and can be higher, i.e. $P_L \geq P^{**}$. Therefore, in contrast with the preemption game, the war of attrition enhances the option's waiting policy.

The exploratory option value depends on the key technical parameter *chance factor* (CF). For each prospect there are three possibilities for the information set in which the oil specialists estimate CF: (a) without (or before) the information revelation from the adjacent prospect, CF; (b) with positive information revelation CF^+ (the neighboring prospect discovers petroleum); and (c) with negative information revelation, CF^- (neighboring drilling find a dry hole). Of course, eq.17 and the boundary conditions (eqs.18) apply for all three cases, just the value of the parameter CF changes. The case (a) is used for the leader L(P, t) valuation, whereas the cases (b) and (c) are used for the informed follower F(P, t) valuation. To evaluate these options we need the relation between CF, CF^+, and CF^-.

Chance factor is a Bernoulli distribution[27], which is estimated by geologists and geophysics using the available information to find the probability of success when drilling an exploratory well. It is a technical uncertainty and therefore it doesn't demand risk-premium by diversified investors.

The relation between the misinformed (before the information revelation from the neighboring drilling) chance factor CF and the informed (or revealed) chance factors CF^+ and CF^- depends on the degree of correlation or association between the prospects. If these prospects are adjacent (low distance) and are place in the same geologic play, this correlation can be very high mainly for the geological synchronism factor (that is typically the lowest factor when we have only seismic information). We can set this correlation degree using the parameter named (proportional) *expected*

[27] Bernoulli distribution is the simplest discrete probability distribution, with only one parameter – the probability of success CF. It has only two scenarios, 1 in case of success and 0 in case of failure, with probabilities CF and (1 – CF), respectively. Of course the mean of this distribution is CF. The variance is the product CF . (1 – CF).

variance reduction (EVR)[28]. For any random variable X and any random signal S both with finite and positive variances, $EVR_{X\,|\,S}$ is defined by the equation below, including a useful relation:

$$\mathbf{EVR_{X\,|\,S} \;:=\; \frac{Var[X] - E[Var\{X\,|\,S\}]}{Var[X]} \;=\; \frac{Var[E\{X\,|\,S\}}{Var[X]}} \qquad \mathbf{(19)}$$

Note that it is directly associated with the *conditional expectation* concept, which is very convenient for value of information applications (see Dias, 2002). The following lemma establishes the information revelation effect on the chance factor CF for a given EVR estimated by the specialists. It is formulated in a generic format but with specific notation in order to ease the context application.

Lemma: Consider two (correlated) Bernoulli random variables, the variable of interest CF_i and the signal CF_j, with parameters also denoted by CF_i and CF_j, respectively, with the latter having strictly positive variance, i.e. $CF_j \in (0, 1)$. The signal's information revelation relevancy is measured by the expected percentage of variance reduction ($EVR_{i\,|\,j}$) over CF_i caused by the signal CF_j. Then the revised CF_i values for both positive revelation and negative revelation are respectively:

$$\mathbf{CF_i^{\,+} = CF_i + \sqrt{\frac{1 - CF_j}{CF_j}}\;\sqrt{CF_i\,(1 - CF_i)\,EVR_{i|j}}} \qquad \mathbf{(20)}$$

$$\mathbf{CF_i^{\,-} = CF_i - \sqrt{\frac{CF_j}{1 - CF_j}}\;\sqrt{CF_i\,(1 - CF_i)\,EVR_{i|j}}} \qquad \mathbf{(21)}$$

In addition, if the Bernoulli random variables CF_i and CF_j are *exchangeable*[29], i.e. $CF_i = CF_j$, the updated CF_i expressions are simplified to:

$$\mathbf{CF_i^{\,+} = CF_i + (1 - CF_i)\,\sqrt{EVR_{i|j}}} \qquad \mathbf{(22)}$$

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[28] It is a measure of association between random variables with nice properties for *value of information* applications (see Dias, 2003), also named *correlation ratio* and attributed to Pearson (see Komolgorov, 1933, p.60). The managerial objective when investing in information is to reduce uncertainty – here best represented by the variance, so that EVR appears at least as one interesting indicator of learning. In addition, the name EVR looks more meaningful for value of information applications than correlation ratio.

[29] N random variables are exchangeable if their joint distribution is the same no matter in which order they are observed. All the variables have the same marginal distribution. It is largely applied in probability and statistics; in particular iid (independent and identically distributed) random variables are exchangeable (but not vice-versa). It is also named *interchangeable* and the theory was developed mainly by de Finetti (see Chow & Teicher, 1997, p.33).

$$\mathbf{CF_i^- = CF_i - CF_i \sqrt{EVR_{i|j}} = CF_i\,(1 - \sqrt{EVR_{i|j}}\,)} \quad \textbf{(23)}$$

With the *revelation spread*, i.e. the updated values difference ($CF^+ - CF^-$) equal to EVR square root:

$$\mathbf{CF_i^+ - CF_i^- = \sqrt{EVR_{i|j}}} \quad \textbf{(24)}$$

Proof: For the equations (20) and (21) apply the Proposition 2 (mean of the revelation distribution, best known as *law of iterated expectation*) and Proposition 3 (variance of the revelation distribution) from Dias (2002) to variables CF_i and CF_j, and the EVR definition (eq.19). That propositions mean here, respectively, that $CF_i = (CF_i^+ \,.\, CF_j) + [CF_i^- \,.(1 - CF_j)]$ (mean preserving property) and $Var\{E[CF_i \mid CF_j]\} = Var[CF_i] - E\{Var[CF_i \mid CF_j]\}$ (expected variance reduction in CF_i). Equations 22, 23, and 24 follow simply by applying of the exchangeable variables definition, i.e. $CF_i = CF_j$.

The <u>leader value</u> $L_i(P, t)$, that is, the value of firm i when exercising the *exploratory* option is:

$$\mathbf{L_i(P, t) = - I_W + CF_i \,.\, R_i(P, t)} \quad \textbf{(25)}$$

In words, the leader spends the drilling investment I_W and obtains the development option $R_i(P, t)$ with probability CF_i (and obtains zero otherwise). The development option $R_i(P, t)$ is calculated with the PDE (eq.15) and its boundary conditions (eqs.16).

The follower value $F_i(P, t)$ for the "free-rider" firm i considers the *expected* information revelation gain with the (leader) firm j option exercise. For firm i, a positive information revelation occurs with probability CF_j and a negative information revelation occurs with probability $(1 - CF_j)$. Hence, the (informed) <u>follower value</u>, that is, the value of firm i with free-rider learning is:

$$\mathbf{F_i(P, t) = CF_j \,.\, E_i(P, t; CF_i^+) + (1 - CF_j) \,.\, E_i(P, t; CF_i^-)} \quad \textbf{(26)}$$

Where $E(P, t; CF_i^+)$ and $E(P, t; CF_i^-)$ are the exploratory option values calculated with parameters CF_i^+ and CF_i^-, respectively. These parameters are calculated with the equations (20) and (21) and the options with the PDE (eq. 17) and boundary conditions (eqs. 18).

For <u>symmetric</u> war of attrition, the (informed) follower will exercise its option in case of positive information revelation (i.e. its option E has the improved parameter CF_i^+) and can wait in case of negative information revelation. This follows because the leader exercises optimally the exploratory option only if it is deep-in-the-money (necessary condition). If the option is deep-in-the-money before the revelation – with CF, it is also deep-in-the-money with CF^+ because $CF^+ > CF$ and the exploratory option is a monotonically increasing function of the parameter CF.

Now imagine a very high initial value for P. For this value the exploratory option can be so deep-in-the-money for both players that the simultaneous exercise is optimal for both players. This occurs when the values of leader and follower become equal, vanishing the incentive to be follower (war of attrition prize = 0). In case of symmetric players, there is a threshold level in which both players must exercise this option for values of P equal or higher, renouncing the information revelation premium. In this *symmetric* game case we have optimal *simultaneous* option exercise when the oil price reaches this level. Denote P_S the *simultaneous exercise threshold*, i.e. the lowest value of P in which L(P, t) = F(P, t). In the *asymmetric* game case we will maintain this notation, but with the meaning that P_{Si} and P_{Sj} are the thresholds above which the strategic interaction does not matter anymore for the firm i and j, respectively. For simplicity, consider initially the symmetric game case. With numerical methods we can solve the equation F(P, t) = L(P, t) obtaining P_S. The value of firm i in the simultaneous exercise is denoted by $S_i(P, t)$ (=$L_i(P, t)$). Formally[30]:

$$\mathbf{P_S(t) = \inf \{ P(t) > 0 \mid L(P, t) = F(P, t), t \in [0, T] \}} \qquad \mathbf{(27)}$$

With the standard convention that the infimum of an empty set is + ∞. One situation where the simultaneous exercise is obviously optimal is when the value of P is so high that the exploratory option is deep-in-the-money even in the negative revelation scenario, that is, even $E(P, t; CF_i^-)$ is deep-in-the-money, so that the information revelation will not change the optimal option exercise. Information is irrelevant for decisions purpose here because the option will be exercised anyway.

Is possible a situation in which $P_S = \infty$ for any t < T? The answer is yes for the full revelation case ($EVR_{i|j}$ = 100%), an extreme case of information revelation. In this case $CF_i^+ = 1$ and $CF_i^- = 0$ so that the information revelation is valuable for any finite value of P, because even with a giant oilfield possibility, a large but finite number multiplied by zero is zero and therefore F(P) > L(P) for any finite value of P (see eqs. 25 and 26, and note that E(.) is strictly increasing in CF). In the full revelation case, learning always is valuable if learning cost is zero as in this free-rider case.

[30] This definition is more adequate than the alternative of setup P_S with an *additional* boundary condition (value matching or smooth pasting) to the PDE (eq. 17) because P_S can be ∞ for extreme cases of learning intensity, as we will see later.

Illustrating the concepts and equations presented, we develop a numerical example for the symmetric war of attrition case with the following parameters[31]:

- Stochastic process (GBM) parameters: $r = \delta = 5$ % p.a.; $\sigma = 15\%$ p.a.; $P(t = 0) = 20$ \$/bbl
- Prospect parameters (symmetric payoffs, i.e. $i = j$): CF = 20%; B = 300 million bbl; q = 15% x exp(– 2 δ); $I_w = 30$ million \$; $I_{DP} = [300 + (2\ B)]$ x exp(– 2 r)
- Other parameters: T = 2 years; $EVR_{i\,|\,j} = EVR_{j\,|\,i} = 10$ %.

With these values, the *development option* (given an exploratory success) becomes deep-in-the-money only when the oil price rises to P* = 26.08 \$/bbl, whereas the *exploratory option* becomes deep-in-the-money only when the oil price rises to P** = 30.89 \$/bbl. So, for oil prices below 30.89 \$/bbl, the waiting policy is the optimal for both players not because the free-rider game but because the real options theory tells that the waiting is better when the exploratory option is not deep-in-the-money, independently of the information revelation gain possibility.

Figure 5 shows the leader and follower curves as well as the thresholds for this numerical example. In the right side of the figure is presented the strategic-form of this option-game for the state variable P = 31 US\$/bbl and T = 2 years. It was inspired and adapted from Smit and Trigeorgis (2004).

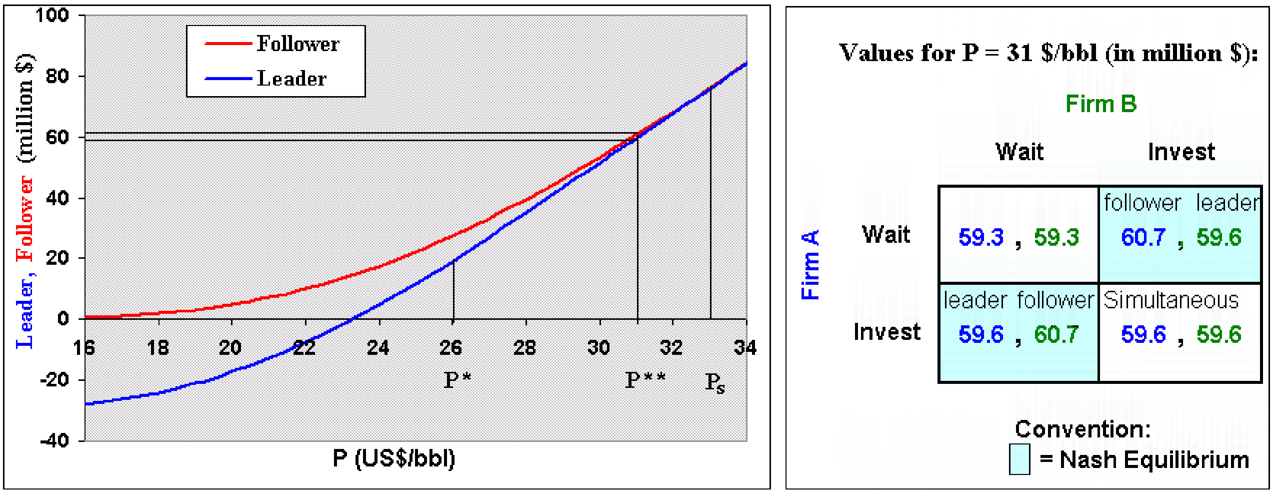

Figure 5 – Leader and Follower in Symmetric Oil Exploration War of Attrition Option Game

[31] The terms exp(– 2 δ) and exp(– 2 r) that appear respectively in the parameters q and I_{DP} are discounting factors: after an oil reserve discovery, the appraisal phase and the development study take about 2 years. Only after that the development option exercise can take place. The discounting factors bring these parameter values from the development option exercise date to the exploratory option exercise date.

This figure eases the discussion of some key points in this war of attrition option-game. First, both leader and follower curves are convex in the underlying stochastic variable (P). Recall from Dias & Teixeira (2003) that in preemption games the leader curve was concave because the increasing demand had offset effects in the leader value, increasing the short-term leader profit but increasing the follower option exercise probability, which decreases the leader market share and so the leader value. In contrast, here the follower exercise does not affect the leader value so that the leader curve has the standard convex option format[32]. The follower value, which is function of the set of possible leader exercise outcomes (information revelation scenarios), is also convex because it is just a convex (linear) combination of two convex functions (standard options functions with different parameters).

Figure 5 also shows the strategic form for the game at the state P(t) = 31 $/bbl and with two years before the option expiration, in order to discuss the pure equilibria features. At this state (and in this subgame) we have two pure strategies Nash equilibrium, (F_i, L_j) and (L_i, F_j), as in the chicken game previously analyzed. In dynamic terms, backward induction shows that if a player is considering to become leader, it is always better to be leader at the first time that the oil price hits the P** level than after this time because there is a fighting cost to delay the deep-in-the-money exploratory option[33]. Hence, the pure MPE strategy for the leader is to choose the stopping time t_L given by:

$$\mathbf{t_L \;=\; inf\,\{\, t \mid P(t) \geq P^{**}(t) \,\}} \qquad \textbf{(28)}$$

The follower pure strategy is to choose a $t_F > t_L$ at the first time that the exploratory option *with the updated information*, becomes deep-in-the-money. Denote the exploratory option threshold after the information revelation of P_F. The stopping time t_F will occur immediately after the information revelation in case of positive revelation and later (or never) if the information revelation is negative.

$$\mathbf{t_F \;=\; inf\,\{\, t \mid P(t) \geq P_F(t) \,\}} \qquad \textbf{(29)}$$

[32] As showed in eq.(25), the leader value is one option (development option) multiplied by a constant (CF) less a constant (I_w), so it is a standard convex option curve with a translation down of I_w. This leader curve only becomes linear in P after P* = 26.08 $/bbl, when the option component R(P, t) becomes deep-in-the-money and linear. In contrast, the traditional *exploratory* option (not showed in Figure 5) only becomes deep-in-the-money (and linear) after P** = 30.89 US$/bbl.

[33] In the right side of Figure 5, the payoff values for the strategy (wait, wait) consider a cost to delay the option exercise for one month with the risk-free interest rate. However, this reasoning is valid for any strictly positive cost of delay so that a more rigorous method to estimate the cost of delaying deep-in-the-money options is not necessary here.

Figure 6 shows a zoom in the previous Figure 5, in order to highlight the interval [P**, P_S), the set of states in which the war of attrition game matters[34].

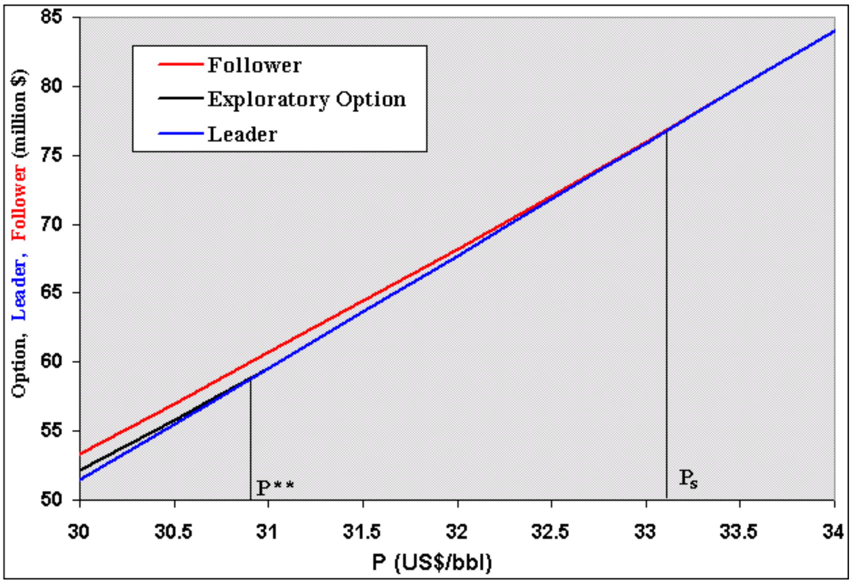

Figure 6 – Zoom from Figure 5 Focusing the Relevant Game Region

Figure 6 also shows the (traditional) exploratory option value, which is higher than the leader value for P < P** (waiting value is the difference), smooth pastes the leader value at P**, and is equal to the leader value when deep-in-the-money (P ≥ P**). At P = 31 $/bbl (> P**) both players have deep-in-the-money exploratory options so that the war of attrition game matters: there is a game incentive to wait (the information revelation prize) and a fighting cost (the cost to delay the exercise of a deep-in-the-money option). For P < 30.89 $/bbl, there is no game (the game does not matter) because the optimal policy ("wait and see") is independent of the presence of other firm. In addition, for P = P_S = 33.12 US$/bbl, the price is so high that the simultaneous exercise is optimal for both players, so that the game is over for P > P_S. For the adopted numerical values, the war of attrition game only exists in the window P ∈ [30.89, 33.12). Of course this interval is small because the information revelation prize is relative small with the adopted value of $EVR_{i\,|\,j}$ = 10%. If we increase the game prize by rising the value of $EVR_{i\,|\,j}$, the war of attrition window will enlarge as well because while the leader value is the same, the more valuable information revelation increases the follower value and so P_S.

[34] So, for the entire remaining region (P < P** and P ≥ P_S) the game does not matter! In a related war of attrition option game paper, Murto (2002) also identifies the state region (window) with no strategic interaction. However, his paper has many differences: application to exit in duopoly; perpetual option (here it is finite); simple abandon option (here is compound option to invest), etc.

Table 1 shows the oil prices window in which the war of attrition matters, using different values of $EVR_{i\,|\,j}$. Note that if $EVR_{i\,|\,j} = 0$ there is no information revelation and hence no game window (empty set), $P_S = P^{**}$ and at P_S the game is over. For $EVR_{i\,|\,j} = 100\%$ – the full revelation case, free cost learning is always optimal and hence $P_S = \infty$, so that for the full revelation case the war of attrition game matters for any deep-in-the-money exploratory option (any $P \geq P^{**}$).

Table 1 – Information Revelation Relevancy ($EVR_{i\,|\,j}$) x Oil Prices in which the Game Matters

$EVR_{i\,|\,j}$ (%)	0	10	20	30	50	70	90	100
Game Window [P, P_S) ($/bbl)**	∅	[30.9, 33.1)	[30.9, 34.8)	[30.9, 36.7)	[30.9, 42.5)	[30.9, 55.5)	[30.9, 120)	[30.9, ∞)

Equilibrium analysis: For the *symmetric* war of attrition game there are two pure strategy perfect equilibria. The first one with firm i exercising its option first at the time t_L and firm j exercising its option at t_F. The second equilibrium is the same but with the firms i and j swapping their roles. The pairs of strategies that are equilibria are (t_{Fi}, t_{Lj}) and (t_{Li}, t_{Fj}). Note that the classical war of attrition result with $t_{Lk} = 0$ occurs if the initial oil price $P(t = 0) \geq P^{**}_k(t = 0)$, k = i, j.

The *asymmetric* war of attrition is a case of more practical interest because is more common. In our application, oil companies have different discount rates, different exploratory prospects (seismic surveys indicate different expected reserves volumes in case of success), different interpretation of the same geological data, etc. Assume that the option *exercise price* I_w is the same for both players[35]. The asymmetry is due to parameters like the expected quality and/or volume of reserves. A known result (see Hammerstein & Selten, 1994, p.978) for asymmetric war of attrition is that typically the benefit-cost ratio V/C should be decisive, i.e. the player with higher V/C (the stronger firm) should win (follower). Here the benefit is the information revelation gain, given by the difference $F_i - L_i$ for the player i. The cost of fighting C_i here depends on how deep-in-the-money is the firm's drilling option and the discount rate, i.e. the cost to delay a deep-in-the-money option exercise per unit of time. The cost to delay an infinitesimal time interval dt, with a risk-free discount rate r, the deep-in-the-money exploratory option that values L_i, is:

$$C_i(P, t) \;=\; L_i(P, t)\ [\,1 - \exp(-\,r\,dt)]$$

[35] Both firms can contract a drilling rig for the same current market daily rate. With the correlated prospects in the same geologic layer, the prospects require approximately the same drilling time and so the same drilling investment I_w. Oil firm asymmetric costs are mainly of operational nature and can be considered in the development phase with the parameter q.

By letting exp(− r dt) ≅ 1 − r dt for a very small dt, and substituting we obtain the following expression for the firm i benefit-cost quotient Q_i:

$$\mathbf{Q_i(P, t) = (F_i - L_i)/C_i = (F_i - L_i)/(L_i\ r\ dt)} \qquad \mathbf{(30)}$$

In option games context, instead of the quotient Q, the firms' asymmetry can be more adequately characterized by the difference of threshold values as in Murto (2002) and in Lambrecht (2001). In our case, the relevant threshold for asymmetric firms is P_S. If the firm i is stronger than firm j, we have $P_{Si} > P_{Sj}$. Recall that at P_S the immediate exercise is optimal independently of the other player. So, the firm with higher threshold P_S is more patient in the sense that firm i can be sure that *before* the oil price reaching the its threshold P_{Si}, the price will reach first the opponent threshold P_{Sj}, so that firm j will exercise before the option. In other words, is not credible the menace of firm j to wait in the interval $[P_{Sj}, P_{Si}]$ so that this Nash equilibrium is not perfect (recall that the perfection criterion points out that the equilibrium needs to be Nash in *all* the possible subgames).

Therefore we have two criteria (benefit-cost quotient and threshold) to setup the asymmetry in this asymmetric game. However, if the risk-free discount rate is the same for both firms, eq.30 tells that the benefit-cost quotient analysis can be reduced to the F/L ratio analysis, i.e. simply the stronger player owns the higher F/L ratio. Given the definition of P_S, in most cases these criteria are equivalent. In order to see this, look the curves F and L in the Figure 6. If we *raise* the leader value curve decreasing the F/L ratio, the threshold P_S decreases as well[36] (for the same I_w). So, these two criteria are equivalent for *finite* P_S. The threshold criterion disadvantage is that for extreme cases of information revelation these thresholds can be *infinite* for both firms ($P_{Si} = P_{Sj} = \infty$) even when there are some differences between the firms' payoffs. In this case the quotient criterion can distinguish the stronger and weaker firms but it is harder to prove a perfect equilibrium, as indicated below.

The asymmetry in war of attritions generally rules out one pure strategy perfect equilibrium, pointing out as *unique* pure strategy equilibrium the intuitive equilibrium with the *weaker* firm exercising its option immediately and the *stronger* one (the more patient) being the follower. Even a small advantage ("ε advantage") is sufficient to make the stronger firm the winner without a fight generating a unique perfect equilibrium. This result is showed in both traditional war of attrition literature (Ghemawat & Nalebuff, 1985; Fudenberg & Tirole, 1991, pp.124-126) and option-game

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[36] In addition, it is intuitive that a more valuable oilfield shall own lower *investment* thresholds P*, P**, and P_S.

war of attrition literature (Lambrecht, 2001; Murto, 2002). Murto (2002) ruled out the "paradoxical" perfect equilibrium with the stronger firm exercising first, either if the degree of uncertainty is from small to moderate – even for small asymmetry, or if asymmetry is sufficiently large in case of high degree of uncertainty. However, Murto (2002) main contribution is to show that for a high enough degree of uncertainty and for sufficiently small asymmetry, the non-intuitive perfect equilibrium with the stronger firm conceding first, can emerge because the exercise threshold is not unique[37]. As in Lambrecht (2001, p.771-772) and for similar reasons (keep simpler *other* extensions), we restrict our analysis to a single exercise threshold P_S for each firm instead disconnected exercise sets.

Hence, for our asymmetric war of attrition the *unique* perfect equilibrium is: the "weaker" firm j (firm with lower P_S) drills the prospect at t_{Lj} (when P reaches P^{**}_j) and the stronger firm i becomes the follower exercising its option at t_{Fi}, the first time that the exploratory option *with the updated information*, becomes deep-in-the-money. The latter (t_F) can occur immediately after t_L in case of positive revelation or, in case of negative revelation, either at the first time that P reaches the updated $P^{**}_i(t)$ or never if the oil price does not reach this level in the interval (t_{Lj}, T]. In more formal terms, the unique perfect equilibrium is the pair of strategies (t_{Fi}, t_{Lj}).

Is this perfect equilibrium stable in ESS sense? According a Selten's theorem, an ESS in the asymmetric game must be a *strict* Nash equilibrium. But as Kim (1993) pointed out, the conjecture of Maynard Smith (1974) that this pure strategy equilibrium can be ESS in *asymmetric* games, can be established even not obeying the Selten's theorem if we replace the ESS by the concept of *limit ESS*, which considers the possibility of "trembles" (or small errors) to players' strategy choices.

However, for the *symmetric* game case in which we have two pure strategy perfect equilibria, (t_{Fi}, t_{Lj}) and (t_{Li}, t_{Fj}), the only candidate to ESS is the *mixed strategy* equilibrium, which is a randomization over these two pure strategy equilibria, contrasting the pure strategy as candidate to be ESS in the asymmetric game. This is a classical Maynard Smith result.

Mixed strategies in this timing game are cumulative probability distribution functions G_i on $t \geq 0$, i.e., $G_i(t)$ is the probability that the player i stops at or before t. The functions $G_i(t)$ don't need be continuous, they can "jump". In their paper on continuous-time war of attrition, Hendricks et al

[37] Murto (2002) calls it "gap equilibrium". In this high volatility case, due to strategic interaction, each firm has at least two state variable regions where the option exercise is optimal independently of the other firm. In *our* case, it means that

(1988) analyze the mixed strategy equilibria, showing that the cumulative distribution G(t) has *concentrated mass points* in equilibrium only at either the beginning or the ending of the game (degenerate mixed strategy equilibria). For *nondegenerate* mixed strategy equilibria they found that under certain conditions there is a continuum of nondegenerate equilibria with positive probability of stopping for both players in the interval (0, t*), i.e. the function G(t) is strictly increasing in this interval, after which both players wait until the game expiration (T) when the function G(t) can jump due to the possibility of mass point at T. However, the same paper points out that for *finite* lived game – mentioning the specific case of oil exploration as example, there is no nondegenerate equilibrium due to the payoffs discontinuity[38] at the game expiration T, with the return to leading strictly exceeding the payoff that can earned at the expiration.

It is interesting to discuss further and more formally the existence or not of mixed strategies in this option-game context[39], as well as the issue of *incomplete information* and the existence of *Bayesian perfect equilibrium*[40]. But for sake of space we will discuss directly a more interesting alternative equilibrium that can Pareto-dominate all these equilibria, Namely the "*changing the game*" alternative, with the players abandoning the war of attrition game in favor of a bargaining game with a win-win binding contract. This is the object of the next section, when we will see the conditions for this contract be better even when compared with the free-rider follower strategy, thanks to the *additional* revelation of *private* information allowed with the partnership, enlarging the game surplus. The problem with the noncooperative solution is that there is mutual gain (surplus) left unexploited.

4 – Changing to Bargaining Game

In this section we discuss mainly the possibility of changing the game from the noncooperative war of attrition game to the *bargaining game*. We follow the advice given by Brandenburger & Nalebuff (1996), who pointed out that in business the biggest profits come from changing the game itself if we are playing the wrong game. In their words, "*changing the game is the essence of business strategy*".

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there is a region between these exercise regions where the stronger player can exercise the option if the price *drops*. So, we have "intermediate waiting regions" similar to the ones found in Dias et al (2003) in traditional real options context.

[38] Hendricks et al (1988) set a different terminal condition with F(T) > L(T) for finite game, instead F(T) = L(T) as here. However, in both cases there are payoff discontinuities at T. Note that F(T) = L(T) is not boundary condition of any PDE here, so that both terminal conditions result in the same value functions and in the same equilibrium analysis.

[39] The nondegenerate equilibria can occur in finite option-games context mainly if we consider the issue of multiple P_S for each firm as in Murto (2002). This analysis can be very complex and it is left for future research.

[40] In the traditional war of attrition literature, Ponsati (1995) proved the existence of a *unique* Bayesian equilibrium in two-player case when are combined two-sided incomplete information and a deadline (finite lived game as here). This

In the oil exploration game, we could enlarge the set of actions by allowing the partnership option. Can a partnership with the firms signing a binding contract be equilibrium? What are the conditions? How to select one from multiple cooperative equilibria?

In order to discuss these points let us present a simple example adapted from Dias (2001). Two firms i and j have equal prospects (same chance factors, same payoffs, etc.) in neighboring tracts. These prospects are in the same geologic play and hence they are correlated. Assume that statistical studies quantify the correlation so that $EVR_{i\,|\,j}$ = 10%. The drilling option is expiring in few days for both firms and the current *expected monetary value* (EMV)[41] of each prospect is negative with the current parameters: for each prospect the chance factor is CF = 30%, the drilling cost is I_w = 30 million \$, and the development NPV in case of exploratory success is NPV_{DP} = 95 million \$. So, the payoff from exercising this expiring exploratory option is:

$$EMV = -\,I_W \;+\; [CF\,.\,NPV_{DP}] \;=-\,30 + [0.3 \text{ x } 95] \;=\; -\,1.5 \text{ million \$}$$

With our previous section model we can assert that both players will not exercise their options and the undrilled tracts will return to the government, so that the value of each firm is zero with these "optimal" strategies in war of attrition game. Let us examine a partnership possibility in order to see if *cooperation*[42] with a binding contract can be best-response strategy for the players.

Imagine the following partnership contract: firms share their prospects with 50% of *working interest* in each prospect for each firm. By this agreement, one well will be drilled immediately and the other one can be drilled or not depending on the information revealed with the first drilling. What are the firm values in this case?

Denote the first drilling well as "well i" and the other one the "well j". First, we need calculate the revised chance factors for the well j in the two scenarios of information revelation, CF_j^+ and CF_j^-. With $EVR_{j\,|\,i}$ = 10% and applying equations 22 and 23, we obtain CF_j^+ = 52.14 % and CF_j^- = 20.51 % so that in case of negative revelation (that occurs with 70% probability) the EMV_j^- is even more

equilibrium is similar to the mixed strategy one pointed before: a function G(t) with a mass point at the expiration and no option exercise on some interval preceding it [t*, T). It is very common that the mixed and Bayesian equilibria coincide.

41 *Expected monetary value* is the equivalent to the *net present value* in exploration business.

42 We use the term *cooperation* instead the term *collusion* used in our previous paper, because the latter has the negative connotation consistent with the negative welfare effect, whereas the binding contract here doesn't penalize the society and it is socially desirable because improves the efficient allocation of resources in Pareto-optimal sense.

negative. However, the well is *optional* and we don't exercise the drilling option in this bad news case. For the case of positive revelation (with chance of only 30%), the EMV_j^+ is:

$$EMV_j^+ = -I_W + [CF_j^+ . NPV_{DP}] = -30 + [0.5214 \times 95] = +19.53 \text{ million } \$$$

Hence, in case of positive revelation from well i, the expiring well j drilling option shall be exercised. The cost to obtain the information revelation is the negative EMV_i from the first drilling. Because the two firms share 50% in both costs and benefits, the value of firm with the *union* of assets U (equal for both firms due to the game symmetry in this example) with the cooperation strategy is:

$$U_i = U_j = 50\% \{EMV_i + [CF_i . Max(0, EMV_j^+)] + [(1 - CF_i) . Max(0, EMV_j^-)]\} \Rightarrow$$

$$\Rightarrow \mathbf{U_i = U_j =} \ 50\% \{-1.5 + [0.3 \times 19.53] + [0.7 \times 0]\} = \mathbf{+2.18 \text{ million } \$}$$

Therefore, both firms earn positive value with this cooperation strategy so that cooperation is a joint best response strategy and perfect equilibrium of this game. The noncooperative alternative – named *disagreement point* (d), has value equal zero ($d_i = d_j = 0$) in this example because in case of disagreement the firms will return the non-drilled tracts back to the Govern. Given the symmetry, it looks natural to choose 50% exchange of assets between the oil companies as the natural equilibrium in this cooperative game. However, there are other cooperative strategies equilibria. In fact, there is a continuum of strategies available that are Nash equilibria, most of them with asymmetric shares. For instance, if the agreement is that firm j share is only 40% (firm i with 60%) of the two assets, the values for the players with this contract are U_i = 2.62 million \$ and U_j = 1.74 million \$. Even being lower than the other player value, U_j is positive so that there is no incentive to unilateral deviation. In practice this is not the more probable equilibrium outcome, but it is also Nash equilibrium. However, our bargaining theory will recommend the 50%-50% share as the *unique* game solution in this case.

Assume that the bargain variable in the binding contract is the joint assets share – or working interest, denoted by w_i for the player i. The set of pair shares $\{w_i, w_j\}$, with $w_j = 1 - w_i$, plus the disagreement point, form the full *feasible set* (***S***) of this bargaining game. For any $w_i \in [0, 1]$, with $w_j = 1 - w_i$, the binding contract $\{w_i, w_j\}$ is simultaneous best response, so that the (noncooperative) Nash equilibria concept cannot help us to select a unique binding contract. However, we can use concepts from *bargaining theory* – a game theory branch, to help us in this equilibria selection job.

We prefer to present the *cooperative bargaining solution* in order to be more comprehensive in the option games illustrations and due to the popularity and good status in the bargaining literature. One alternative to the cooperative bargaining theory is the *noncooperative bargaining theory*, with alternating offer and counteroffer and using backward induction to determine the game solution. However, there is a close link between these two approaches: if we allow a small risk of breakdown after any rejection to an offer, the noncooperative bargaining solution converges to the Nash's *cooperative* bargaining solution as the breakdown probability goes to zero (see Osborne & Rubinstein, 1994, section 15.4). We'll exploit further this issue in order to get a *perfect* equilibrium. A third and more recent alternative is the *evolutionary bargaining theory*; see for example Napel (2002). The cooperative approach is easier to apply because it is independent of the negotiation framework, whereas the noncooperative one is very sensible to the extensive form specification.

Given a cooperative game defined by the feasible set and the disagreement point, the pair ($\boldsymbol{S}$, d), with $\boldsymbol{S}$ convex, bounded and closed, and with at least one point strictly dominating $d \in \boldsymbol{S}$, the three most acceptable criteria to select a *solution*[43] are[44]: (a) the Nash (1950) solution, which recommends the point of $\boldsymbol{S}$ at which the product of payoff gains from d is maximal; (b) the Kalai & Smorodinsky (1975) solution, which suggests the point of $\boldsymbol{S}$ so that the payoff gains from d are proportional to their maximal possible values inside the subset of feasible points dominating d; and (c) the egalitarian solution, which recommends the solution that equates payoff gains from d. In our previous simple *symmetric* example, the natural solution $w_i = w_j = 50\%$ coincide for all the three criteria. For the case of asymmetric payoffs (e.g., $d_i = 0$ but $d_j = + 1$ million \$), this coincidence does not hold.

The Nash's cooperative solution is an *axiomatic* approach based *mainly* on three general principles: (a) *scale invariance*, i.e. the solution does not change in case of *linear* transformations in the payoff scale; (b) *outcome efficiency*, i.e. the bargainers obtain summed no less than the full available surplus (no mutual gain is left unexploited); and (c) *contraction independence*, i.e. the solution is invariant to removal of irrelevant (non-adopted) feasible alternative solutions. Nash (1950) proved that his solution is <u>the unique</u> solution satisfying the axioms of scale invariance, Pareto-optimality (a stronger efficiency criterion), contraction independence, and *symmetry*[45]. However, the Nash solution formulation without the symmetry assumption became more common in economic applications

[43] *Solution* is a rule that gives the proportions of division of surplus.
[44] See Thomson (1994) for an advanced but concise discussion on these and other cooperative bargaining solutions. For a nice introduction to Nash's cooperative solution at elementary level, see Dixit & Skeath (1999, ch.16).
[45] The bargaining problem ($\boldsymbol{S}$, d) is symmetric if the solution $f_i(\boldsymbol{S}, d)$ with one player is equal with the other player $f_j(\boldsymbol{S}, d)$.

(Dixit & Skeath, 1999, p.528) even generating multiple solutions (a degree of freedom to consider some other variable in the solution selection). We do consider the Nash's solution with the symmetry axiom in order to obtain a unique solution in our option game application.

In this bargaining game, define U as the union of the assets from firm i and j. The agreement shares are denoted by $\mathbf{U_i = w_i \,.\, U}$ and $\mathbf{U_i = w_j \,.\, U}$, with $\mathbf{w_j = 1 - w_i}$ and these weights being calculated with the Nash´s axiomatic rule. The value of the assets union U is given by the EMV of one prospect (that can be negative, recall the example) plus the expected value of the other prospect updated with the information revelation from the first drilling (always positive). But a necessary condition for the firms´ agreement is $U \geq 0$ (partnership is an option, not obligation). Hence,

$$\mathbf{U = Max\{\, 0, EMV_j + [CF_j \,.\, E_i(P, t; CF_i^{+})] + [(1 - CF_j) \,.\, E_i(P, t; CF_i^{-})] \,\}} \qquad \mathbf{(31)}$$

In the above equation we assume that the prospect j will be drilled first to reveal information. Recall that in the asymmetric case the prospect j is the "weaker" one and shall be drilled first. In the bargaining contract, firms will split U according the Nash´s solution. **Figure 7** illustrates the Nash´s solution in this cooperative bargaining game under uncertainty.

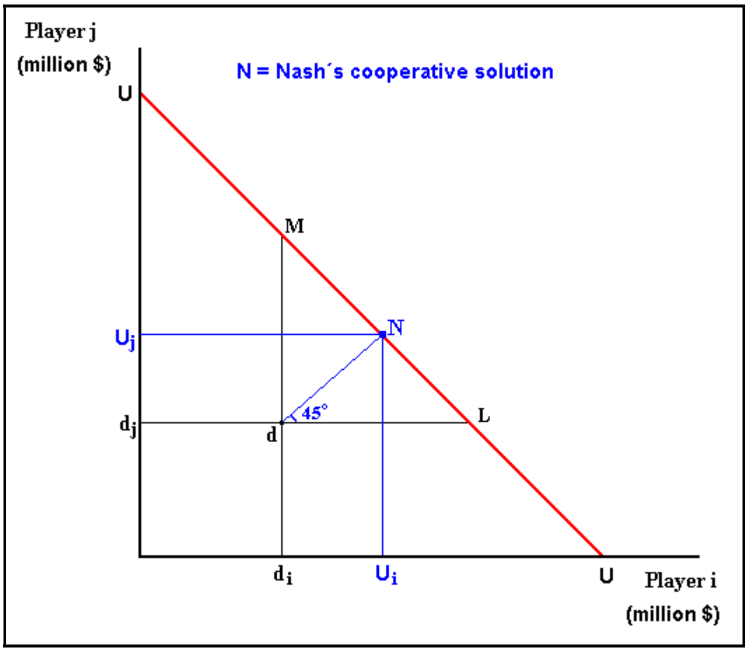

Figure 7 – The Nash´s Cooperative Bargaining Solution

In Figure 7, the red line represents the feasible set of agreements (all convex combinations splitting U) and the disagreement point d with coordinates (d_i, d_j) represents the value of firms in case of no partnership (this key point is better discussed below). Note that only the red line segment L-M is of interest, because otherwise firms are better off with the disagreement point payoffs. The point N (in blue) in this segment is the unique Nash´s cooperative bargaining solution. The symmetry axiom simply means that the segment d-N has 45^{o} slope, whereas without this symmetry axiom we could choose any other point from the L-M segment. It is easy to deduce the following equations that characterize the Nash's bargaining solution:

$$\mathbf{w_i = \tfrac{1}{2} + (d_i - d_j)/(2\,U)} \quad , U > 0 \text{ and } w_i \in (0, 1) \qquad \textbf{(32a)}$$

$$\mathbf{w_j = \tfrac{1}{2} - (d_i - d_j)/(2\,U)} \quad , U > 0 \text{ and } w_j \in (0, 1) \qquad \textbf{(32b)}$$

$$\mathbf{U_i = w_i \,.\, U} \quad , U > 0 \text{ and } w_i \in (0, 1) \qquad \textbf{(32c)}$$

$$\mathbf{U_j = w_j \,.\, U} \quad , U > 0 \text{ and } w_j \in (0, 1) \qquad \textbf{(32d)}$$

The link between the cooperative and noncooperative bargaining theories – named "Nash program", has been discussed since Nash (1953) with his concept of *threat game*. More recent research has showed that, for a wide range of cases of practical interest, the noncooperative bargaining game *perfect equilibrium* (unique in some cases) converges to the Nash cooperative solution[46]. It enhances the relevancy of Nash bargaining solution. A novelty issue in this paper with the "changing the game" approach is that the disagreement point comes from the noncooperative option-game[47], i.e. the <u>war of attrition perfect equilibrium enters as input in this cooperative bargaining game</u>. We follow Binmore & Rubinstein & Wolinsky (1986) in that the Nash bargaining solution is a perfect equilibrium in the analogous noncooperative bargaining game of alternating offers, under the assumption of a small probability of breakdown converging to zero and with the proper disagreement point choice. The war of attrition outcome is <u>not</u> an *outside option* in the negotiation table. It is the undesirable Pareto-inferior outcome from the disagreement game event[48]. Under the assumption that only perfect Nash equilibria are credible threats in the disagreement game that follows the bargaining

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[46] Binmore (1987) was the first to prove this relation by allowing that the time interval between the bargaining offers tend to zero. Binmore & Rubinstein & Wolinsky (1986) consider the case of risk of breakdown (as here). Rubinstein & Safra & Thomson (1992) extend this relation for the more general case of *non-expected* utility preferences.

[47] The game sequence is as follows. First firms are playing a noncooperative game (war of attrition) when one or both firms identify a Pareto-superior gain with a bargaining game. Firms change the game by starting the bargaining game. With some (very high) probability p they agree a sharing rule for the union of assets and with probability (1 – p) they disagree. In the latter case the only alternative is to play the disagreement game, the noncooperative war of attrition.

[48] Bargaining breakdown is a random event with small probability. It can occur in case of change of manager, or simply with the passage of time due to a change of state (P, t) in a way that the bargaining alternative becomes less attractive.

game in case of breakdown, perfection requires that the disagreement point be perfect equilibrium in the war of attrition disagreement game. Our combination of war of attrition and Nash bargaining solution in general is not the Nash's threat game[49]. We consider that is not possible to commit *credible* threats other than perfect equilibria from the disagreement game[50]. So, our solution for the bargaining game is the Nash solution with the (best refined) perfect Nash equilibrium from the war of attrition game.

If this noncooperative game has a unique asymmetric equilibrium pointed out in the last section, (t_{Fi}, t_{Lj}), then the disagreement point coordinates are ($d_i = F_i$, $d_j = L_j$). Of course we can use as disagreement point the paradoxical equilibrium (t_{Li}, t_{Fj}) or even a Bayesian or mixed strategy equilibria, if they exist. So, it is a flexible and rich approach for further research.

The bargaining game alternative has an important advantage over the war of attrition in oil exploration because can exploit the entire potential of information surplus, i.e., we can obtain Pareto optimal outcomes with the bargaining game alternative. Public information is only a subset of the information accessible to the partners. In comparison with the public information revelation modeling presented in section 3, cooperation can enhance the information revelation effect in the chance factors (additional private information)[51] and can provide some useful information revelation on the volume and quality of the possible oil reserve[52]. To keep simple, let us consider the effect in the variable EVR alone by making EVR | public information < EVR | private (cooperative) information. Denote the EVR with private information by EVR^*.

Consider the example presented in the last section. Recall that the information revelation with public externality is $EVR_{i\,|\,j} = EVR_{j\,|\,i} =$ 10 %. With the richer information revelation that is expected to be obtained with the partnership, assume that $EVR^*_{i\,|\,j} = EVR^*_{j\,|\,i} =$ 30 %.

In this example, instead choosing equilibrium like ($d_i = F_i$, $d_j = L_j$) for the disagreement point, we will work with a fictitious and extreme disagreement point ($d_i = F_i$, $d_j = F_j$). Imagine that both players think themselves as the stronger player one in the war of attrition. If even in this case the bargaining

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[49] See the difference between disagreement and threat games in Binmore (1992, pp.261-265 and ex.7.9.5d, p.331).
[50] But in case of multiple Nash equilibria in the disagreement game, the Nash threat game could make sense because threat strategies are equilibria and hence credible threats. Bolt & Houba (1998) presented a model where all threats are Nash equilibrium in the disagreement game and each (credible) threat is a disagreement point in the Nash threat game.
[51] Detailed private information can confirm the geologic synchronism with the first drilling, increasing the chance factor CF in the neighboring tract, even with negative *public* information revelation.

alternative is more valuable, then the bargaining alternative will dominate all the possible equilibria that can be inputted as disagreement point. This is very important in practice because we save time avoiding the analysis of irrelevant alternative equilibria, if the bargaining alternative is dominant for the extreme favorable case from the noncooperative war of attrition. Only in case of no dominance with this fictitious bargaining alternative, is necessary to study realistic noncooperative equilibria to input the bargaining game with less favorable disagreement points. However, it only guarantees that in a certain oil prices interval the bargaining game has supremacy over the war of attrition, not the best bargaining solution. The bargaining solution using the input ($d_i = F_i$, $d_j = F_j$) is not the more adequate *bargaining* solution for firms agreeing the existence of asymmetry, e.g. seismic surveys indicating a bigger oil reserve for the firm j, but this solution remains belonging to the Pareto efficient feasible set in the asymmetric problem if we reduce d_i and/or d_j in any other equilibrium. In addition, the bargaining solution with ($d_i = F_i$, $d_j = F_j$) remains strictly higher than the best war of attrition outcome (F) for each player.

We define $\underline{\mathbf{P}}_U$ as the lowest oil price in which the bargaining alternative is *not* inferior to *any* outcome from the alternative war of attrition game. Similarly, we define $\overline{\mathbf{P}}_U$ as the highest oil price in which the bargaining alternative is strictly better than the best war of attrition outcome. Formally, for the player i (for the player j is similar), these "changing the game" thresholds are defined by:

$$\underline{\mathbf{P}}_U(\mathbf{t}) = \mathbf{inf}\{\mathbf{P(t)} \mid \mathbf{U_i(P, t) > 0, U_i(P, t) \geq F_i(P, t)}\} \quad \mathbf{(33a)}$$

$$\overline{\mathbf{P}}_U(\mathbf{t}) = \mathbf{sup}\{\mathbf{P(t)} \mid \mathbf{U_i(P, t) > 0, U_i(P, t) > F_i(P, t)}\} \quad \mathbf{(33b)}$$

These two "changing the game" thresholds form the bargaining game threshold window $[\underline{\mathbf{P}}_U, \overline{\mathbf{P}}_U]$ in which the bargaining game dominates any alternative war of attrition game[53]. Recall the latter has a game window (when the war of attrition is relevant) of $[P^{**}, P_S)$. We say that the bargaining option game *dominates* the war of attrition option game if $[P^{**}, P_S) \subset [\underline{\mathbf{P}}_U, \overline{\mathbf{P}}_U]$.

Figure 8 presents the firm value with the bargaining alternative ($U_i = U_j$), the follower value curve, and the leader value curve versus the oil prices, two years before the option expiration. Although there is a range of oil prices in which the follower value is higher than the bargainer value, we note

[52] Nonpublic information details like the *oil-water contact* depth and the fluid/rock properties found with the first drilling, are useful to update the expected volume and quality of the reserve of the neighboring prospect.

that the bargainer value is always higher than the leader value (except for very high oil prices, when they are equal) and equal or higher than the follower value for oil prices in the option game window $\underline{P}_U$ = 29.6 \$/bbl and $\overline{P}_U$ = 36.7 \$/bbl. So, the bargaining game window contains the war of attrition window [P**, P_S) = [30.89, 33.12) – see the section 3, and the former full dominates the latter.

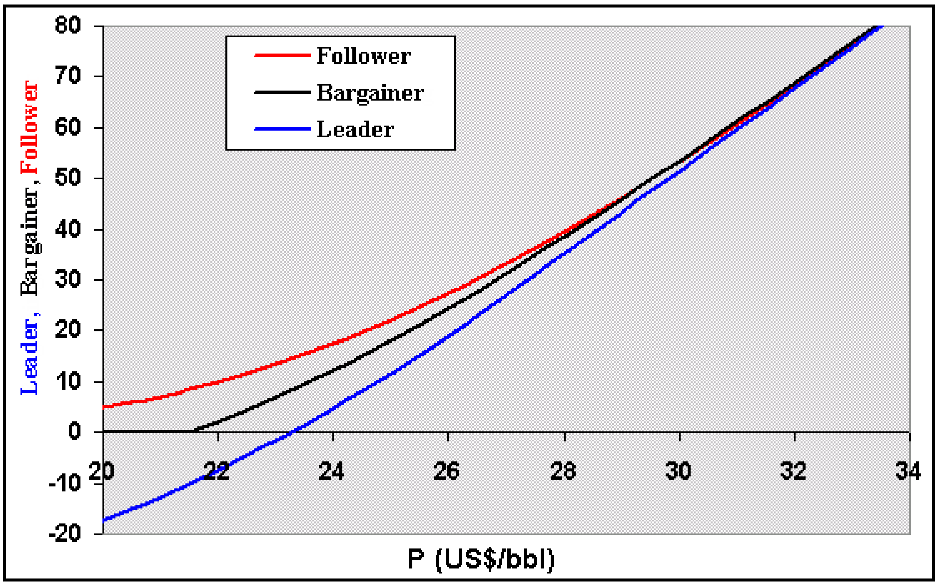

Figure 8 – Bargaining and War of Attrition Joint Game Analysis

Figure 9 shows a zoom from Figure 8 to highlight the interval in which the war of attrition matters [30.89, 33.12). Note that the bargainer value is always higher than the best war of attrition outcome (follower value), thanks to the addition information revelation obtained with a private contract. Note also that the difference between the bargainer and leader values is strictly decreasing with the oil price so that is intuitive that they meet for very high oil price (really they meet at $\overline{P}_U$ = 36.7 \$/bbl).

[53] We are assuming this interval is unique, which we believe is true for most practical cases.

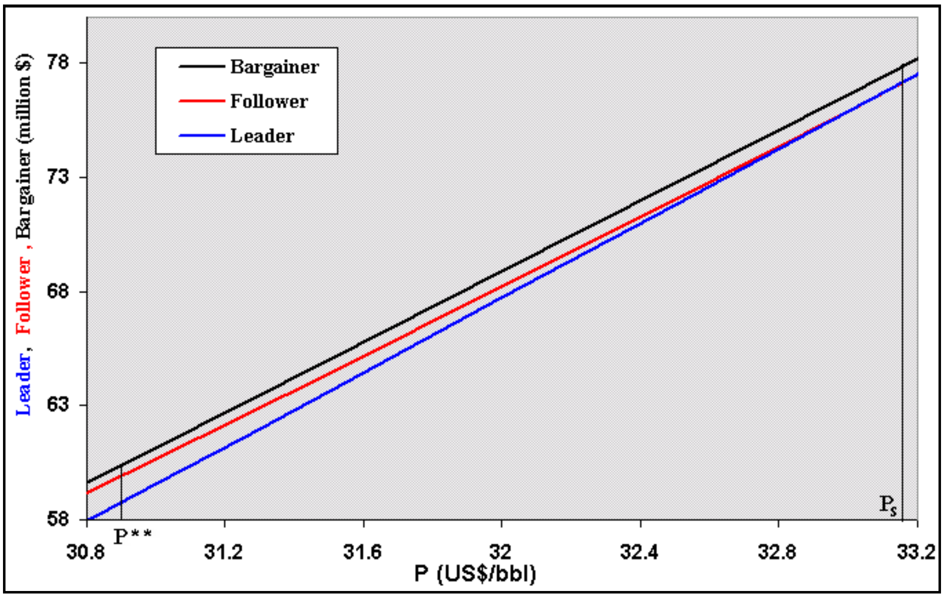

Figure 9 - Zoom from Figure 8 Focusing a Dominating "Changing the Game" Region

If t_U is the first time in which the oil price reaches the threshold $\underline{\mathbf{P}}_U$, note that it is <u>not</u> the first time in which is optimal to change the game from war of attrition to bargaining game. The changing the game can be optimal before t_U, if in the disagreement point we consider lower war of attrition equilibrium payoffs replacing the extreme case of the free-rider follower payoff. However, for P in the interval $(\underline{\mathbf{P}}_U, \overline{\mathbf{P}}_U)$ we guarantee that the bargaining game dominates strictly the war of attrition one for any noncooperative equilibrium outcome.

What about the option game premium? In both war of attrition and bargaining games this premium is higher than the (non-strategic) real option premium. In the first case because there is an additional incentive to wait due to the higher value that can be obtained by the follower with the information revelation spillover. In the bargaining game, this premium is higher because the binding contract permits both firms to exploit the entire information surplus, which in general is higher than the public information surplus disputed by the war of attrition players, thanks to additional private information.

5 - <u>Conclusions and Suggestion to Future Research</u>

In this paper we presented the model of oligopoly under uncertainty from Grenadier (2002) with discussion of concepts like the Leahy's "*optimality of myopic behavior*", the change in demand function in order to solve oligopoly as an *artificial perfect competitive market*. In addition, we reviewed models of positive externalities focusing on war of attrition models – extending the case discussed in Dias (1997), and bargaining game model.

We saw that the *option premium* for option games models can be *negative* (in some cases of preemption, as in Huisman & Kort, 1999, see our former paper), positive but *approaching zero* when the number of competitors grows (oligopoly, Grenadier, 2002), zero (perfect competition, infinite firms oligopoly), or even *higher* than the standard real option premium (as presented here for war of attrition and bargaining games). The option to drill the wildcat is more valuable in the option game framework when working with the information revelation possibility, than the value obtained with the real options perspective alone. In terms of *decision rule*, the game theoretic insight can either enlarge (some cases in war of attrition game) or reduce (preemption game and in some cases of bargaining game) the expected time to exercise the option.

We show that cooperation by changing the game from war of attrition to bargaining game can be perfect equilibrium depending on the difference of the information revelation intensity when compared with the (free rider) follower strategy. In this changing the game framework, the noncooperative war of attrition perfect equilibrium enters as input (the disagreement point) in the cooperative bargaining game. We set a state space (oil prices) window in which bargaining game dominates any alternative war of attrition game outcome. We work an example that this bargaining game window contains the entire interval where the war of attrition could matter, so that in many practical cases we can expect that the bargaining game full dominate the war of attrition.

Surprisingly, this changing the game approach (or similar) has not been analyzed before in the option game literature at the best of our knowledge. However, the great practical appeal for business decisions together with the interesting theoretical appeal (combination of two game models), indicate the compelling necessity to study these possibilities in both theoretical level and practical applications.

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